This examination consists of two parts, A and B. Part A contains five problems of which you must select four to do. Part B contains three problems of which you must select two to do. Each problem in part A is worth 15 points and each problem in part B is worth 20 points. Only hand-in your solutions to four problems from part A and two from part B. Begin each problem on a new sheet of paper and be sure to label each page of your work with the problem number and your name. You have two hours to complete part A and one hour and 20 minutes to complete part B. There will be ten-minute break between parts A and B.

In each question, if you appeal to a theorem within your solution, you must carefully state that theorem. All graphs, unless otherwise stated, should be understood to be finite and simple.
Problem A1: Let $G$ be a graph in which any two odd cycles intersect.
   a) Prove that $G$ is 5-colorable.
   b) Give an example to show that 4 colors do not suffice.

Problem A2: Find all 3-regular plane graphs in which all faces are triangles. Prove your list is complete.

Problem A3: Let

$$S(n) = \{ (A, B) : \emptyset \subseteq A \subseteq B \subseteq \{1, 2, \ldots, n\}\}.$$  

   a) Find a recurrence relation for $a_n = |S(n)|$.
   b) Find a compact formula for $|S(n)|$. Justify your answer.

Problem A4: Prove that if every chain and every antichain of a poset $P$ is finite, then $P$ is finite.

Problem A5: Consider the ways to distribute $n$ identical balls to five different boxes with the first four boxes receiving between 3 and 8 balls.
   a) Write the ordinary generating function for the number of these distributions.
   b) In how many ways can 25 balls be distributed in this way?
Problem B1: Prove the given identity:

a)
\[ \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1} - 1}{n+1} \]

b)
\[ \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \]

Problem B2:

da) State the definition of what it means for a graph to be an interval graph.
b) State the definition of what it means for a graph to be a perfect graph.
c) Prove directly that every interval graph is perfect.

Problem B3: Use inclusion-exclusion to find a formula for the number of 1-factors in the graph obtained from \( K_{n,n} \) by removing the edges of a perfect matching.