

Instruction: You are required to complete this examination within 3 hours. Do only 5 problems, at least 2 from each group. Work on one side of the page only. Start each problem on a new page. In order to receive full credit, answer each problem completely and must show all work.

GROUP A

1. If $A_1, A_2, \dots, A_n, \dots$ is a sequence of events in a sample space S such that $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq \dots$, then prove that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} P(A_n).$$

Why this result is called the continuity theorem for the probability measure?

2. Consider the function

$$f(x, y) = \begin{cases} 2 & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that f is a bivariate density function.

Let the random variables X and Y have f as their joint density function.

Determine

(b) the marginal density, $f_1(x)$, of X ;

(c) the conditional density, $f(y/x)$, of Y given that $X = x$;

(d) the mean of this conditional density, that is $E(Y/X = x)$;

(e) the variance of this conditional density, that is $Var(Y/X = x)$.

3. Answer any four from the followings:

(a) State precisely Chebychev's inequality.

(b) State precisely the weak law of large numbers.

(c) State precisely Kolmogorov's inequality.

(d) State precisely the strong law of large numbers.

(e) State precisely the central limit theorem.

4. Answer the followings:

(a) Suppose X is a nonnegative random variable with mean $E(X)$. Then show that

$$P(X \geq t) \leq \frac{E(X)}{t}$$

for all $t > 0$.

(b) Suppose X_1, X_2, \dots is a sequence of random variables defined on a sample space S . Define the notion of convergence in probability of the sequence random variables $\{X_n\}$ to X .

(c) Suppose the random variable X and the sequence X_1, X_2, \dots , of random variables are defined on a sample space S . Define the notion of convergence almost surely of the sequence random variables $\{X_n\}$ to X .

(d) Suppose X is a random variable with cumulative density function $F(x)$ and the sequence X_1, X_2, \dots of random variables with cumulative density functions $F_1(x), F_2(x), \dots$, respectively. Define the notion of convergence in distribution of the sequence random variables $\{X_n\}$ to X .

GROUP B

5. Answer the followings:

(a) What is the basic principle of the maximum likelihood estimation?

(b) Define an estimator for a population parameter θ .

(c) Define an unbiased estimator of a parameter θ .

(d) Define a sufficient estimator of a parameter θ .

(e) Let X_1, X_2, \dots, X_n be a random sample from a normal population with known mean μ and unknown variance $\sigma^2 > 0$. Determine the maximum likelihood estimators of μ and σ^2 ?

6. Observations y_1, y_2, \dots, y_n are assumed to come from a model with $E(Y_i/x_i) = \alpha + \beta x_i$, where α and β are unknown parameters, and x_1, x_2, \dots, x_n are given constants. Derive the least squares estimates of the parameters α and β in terms of the sum of squares S_{xx} and S_{yy} , and the sum of cross product S_{xy} , where

$$S_{xy} = \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}).$$

7. Answer the followings:

- (a) Define a $100(1 - \alpha)\%$ confidence interval for a parameter θ .
- (b) Define an interval estimator of a parameter θ .
- (c) Define a pivotal quantity for a parameter θ .
- (d) If X_1, X_2, \dots, X_n is a random sample from a population with density

$$f(x; \theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is an unknown parameter, then derive a $100(1 - \alpha)\%$ approximate confidence interval for θ assuming the sample size is large.

8. Answer the followings:

- (a) Define a hypothesis test.
- (b) Define the power function of a hypothesis test
- (c) State precisely the Neyman-Pearson Lemma.
- (d) Let X_1, X_2, \dots, X_n be a random sample from a normal population with mean μ and known variance σ^2 . Derive the likelihood ratio test of size α for testing the null hypothesis $H_o : \mu = \mu_o$ versus the alternative hypothesis $H_a : \mu \neq \mu_o$?