Combinatorics Qualifying Exam
October, 2005

This examination consists of two parts, A and B. Part A contains **six** problems of which you must select four to do. Part B contains **three** problems of which you must select two to do. Each problem in part A is worth 15 points and each problem in part B is worth 20 points. Only hand-in your solutions to four problems from part A and two from part B. Begin each problem on a new sheet of paper and be sure to label each page of your work with the problem number and your name. You have two hours to complete part A and one hour and 20 minutes to complete part B. There will be ten-minute break between parts A and B.

In each question, if you appeal to a theorem within your solution, you must carefully state that theorem. All graphs, unless otherwise stated, should be understood to be finite and simple.

Part A: **(15 points each)** *Do any four.* **Time: 2 hours.**

Problem A1: An *orientation* of a graph G is a digraph D obtained by inserting an arrow on each edge of G. Prove that a graph G always has an orientation D such that

$$|\deg_D^+(v) - \deg_D^-(v)| \leq 1,$$

for every vertex v of G.

Problem A2: Acme Airlines has n different routes (numbered 1 through n). A schedule set in advance gives the starting time s_i and the finish time f_i for each route i. Let t_{ij} be the time required to move an airplane from destination route i to the origin or route j. This partially orders the routes: place (i, j) in the partial order P if and only if $f_i + t_{ij} < s_j$; that is, routes i and j are comparable iff a single plane can run both routes.

a) State Dilworth's Theorem.

b) What is the minimum number of planes needed to fly Acme's routes?

Problem A3: Consider the alphabet $X = \{a, b, c\}$. Let w_n denote the number words (sequences) of length n over the alphabet X in which the letter b appears an even number of times.

a) Find the exponential generating function for w_n.

b) Find a compact formula for w_n.

Problem A4: The integer 3 can be expressed as an ordered sum of positive integers in four ways, namely, 3, $2+1$, $1+2$, and $1+1+1$. Prove that any positive integer n can be expressed as an ordered sum of positive integers in 2^{n-1} ways.

Problem A5: A subset of the set $\{1, \ldots, n\}$ is *alternating* if its elements, when arranged in increasing order, follow the pattern: odd, even, odd, etc. For example, $\{3\}$, $\{1, 2, 5\}$, and $\{3, 4\}$ are alternating subsets of $\{1, 2, 3, 4, 5\}$, whereas $\{2, 3, 4, 5\}$ and $\{1, 3, 4\}$ are not. The empty set is considered alternating. Let a_n denote the number of alternating subsets of $\{1, \ldots, n\}$.

a) Find a recurrence for a_n.

b) Solve the recurrence in part (a) and find a formula for a_n.

Problem A6: The n-cube Q_n (for $n \geq 1$) is the graph whose vertices are the binary words of length n and two vertices are joined by an edge if and only if their corresponding binary words differ in exactly one coordinate. Show that Q_n is planar if and only if $n \leq 3$.

Part B: **(20 points each)** *Do any two.* **Time: 1 hour and 20 minutes.**

Problem B1: Evaluate the given sum. Justify your answer:

a)

$$\sum_{r=0}^{n} 2^r \binom{n}{r}$$

b)

$$1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}.$$

Problem B2:

a) Prove that if G is a 3-regular simple graph then the vertex-connectivity of G is equal to the edge-connectivity of G.

b) Prove that if G is a simple graph on n vertices with minimum degree $\delta \geq \frac{n+k-2}{2}$, then G is k-connected.

Problem B3: An $r \times s$ *Latin rectangle* based on $1, \ldots, n$ is an $r \times s$ matrix such that each entry is one of the integers $1, \ldots, n$ and each integer occurs in each row and column at most once. Prove that every $r \times n$ Latin rectangle based on $1, \ldots, n$ can be extended (by adding rows) to an $n \times n$ Latin square. (Hint: Do induction on r. Create an appropriate bipartite graph and show the existence of a perfect matching in it to extend the Latin rectangle).