5/13/05                                Spring Analysis Qualifier

Name:

The test consists of three parts.

1. The first part consists of 5 True/False questions. This part is mandatory.

2. You will be awarded credit for 3 out of 5 questions in the second part, and 3 out of 5 questions in the third part. In the following boxes, please circle the 3 that you would like us to grade.

Please do all of the following:

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<th>Part I</th>
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Note: Please note that a complete and correct solution will carry far more weight than several sparsely supported "solution sketches".

FINAL SCORE (out of 175):          PASS          FAIL
PART I. State whether each of the following statements is True (T) or False (F). Support your assertion with a proper justification. You will receive 2 points for the correct choice and 3 points for the justification.

(a) There exists a monotone function $f : [0, 1] \to [0, 1]$ such that $f$ is discontinuous precisely at points in the Cantor ternary set. 
   T   F

(b) Let $\{f_n\}$ be a sequence of non-Riemann integrable functions on $[0, 1]$. If $f_n \to f$ uniformly on $[0, 1]$, then $f$ is also not Riemann integrable. 
   T   F

(c) There is a dense open subset $A$ of the reals, such that $A$ has an uncountable complement. 
   T   F

(d) There exists a monotone function $f : [0, 1] \to [0, 1]$ such that $f$ is non-differentiable precisely at points in a Cantor set of positive Lebesgue measure. 
   T   F

(e) Every closed and bounded subset of a complete metric space is compact. 
   T   F
PART II.  1. Clearly state the Cauchy-Schwarz inequality. Show that if \( \sum_{n=1}^{\infty} a_n^2 \) converges absolutely, then \( \sum_{n=1}^{\infty} \frac{a_n}{n} \) must also converge absolutely.

2. Clearly state the Heine-Borel Theorem. For a bounded open set \( A \) of real numbers, give an explicit construction of an open cover that has no finite subcover.


4. Clearly state the Riesz Representation Theorem for \( L^p[0, 1] \) for \( 1 < p < \infty \). For \( g \in L^1[0, 1] \), prove that \( T(f) = \int_0^1 f g \, dx \) defines a bounded linear functional on \( L^\infty[0, 1] \) and find \( ||T|| \).

5. For a continuous function \( f \) on \([0, 1]\), clearly define its Riemann integral as a limit of partial sums using uniform partitions. Show that
\[
\lim_{n \to \infty} \left( \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} \right) = \ln 2.
\]
PART III. 1. Suppose $f \in \mathcal{L}^1[0,1]$ and set
\[ F(x) = \int_0^x f(t) dt. \]
Prove that $F$ is of bounded variation on $[0,1]$.

2. Let $f : \mathbb{R} \to \mathbb{R}$. Show that $f$ is discontinuous on a set of first category in $\mathbb{R}$ if and only if $f$ is continuous at a dense set of points.

3. Let $\mathcal{H}$ be a separable Hilbert space, and let $\{\phi_k\}$ be an orthonormal set in $\mathcal{H}$. Show that for any $x \in \mathcal{H}$, we have
\[ \sum_{k=1}^{\infty} | \langle x, \phi_k \rangle |^2 \leq ||x||^2. \]
State a sufficient condition for equality.

4. Suppose that $\nu$ is a $\sigma$-finite signed measure and $\mu$ is a $\sigma$-finite measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$. Show that if $g \in \mathcal{L}^1(\nu)$, then
\[ g \frac{d\nu}{d\mu} \in \mathcal{L}^1(\mu) \text{ and } \int g d\nu = \int g \frac{d\nu}{d\mu} d\mu, \]
where $\frac{d\nu}{d\mu}$ is the Radon-Nikodym derivative of $\nu$ with respect to $\mu$.

5. Let $C$ denote the Cantor ternary set in $[0,1]$. Show that
\[ C - C = \{a - b \mid a, b \in C\} = [-1,1]. \]