1. For systems with the potential energy function \( V(r) \) depending only on the distance \( r \), the wave function can be expressed as the product of the radial wave function \( R_{nl}(r) \) and the spherical harmonics \( Y_{lm}(\theta, \phi) \) where \( Y_{lm}(\theta, \phi) \) is the common eigenfunction of operators \( L^2 \) and \( L_z \) such that \( L^2 Y_{lm} = l(l+1) \hbar^2 Y_{lm} \) and \( L_z Y_{lm} = m \hbar Y_{lm} \). (a) Show that the radial wave equation is given by
\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR_{nl}}{dr} \right) + \left( \frac{2m}{\hbar^2} [E - V(r)] - \frac{l(l+1)}{r^2} \right) R_{nl} = 0.
\]
(b) By making the substitution \( R_{nl} = u_{nl}(r)/r \), show that the radial wave function can be reduced to an effective one-dimensional Schrödinger equation
\[
d^2u_{nl}/dr^2 + (2m/\hbar^2)[E - V_{\text{eff}}]u_{nl} = 0 \quad \text{where} \quad V_{\text{eff}} = V(r) + l(l+1)\hbar^2/2mr^2.
\]
(c) For the case of the isotropic harmonic oscillator with \( V(r) = m\omega^2 r^2/2 \), the effective one-dimensional radial wave equation can be written as
\[
d^2u_{nl}/d\rho^2 + (C - \rho^2 - l(l+1)/\rho^2)u_{nl} = 0
\]
with the introduction of \( \rho = \theta r \), \( \alpha = (m\omega/\hbar)^{1/2} \), and \( C=2E/\hbar\omega \). By examining its asymptotic solutions at \( r \to 0 \) and \( r \to \infty \) respectively, show that \( u_{nl} \) can be written as \( u_{nl}(\rho) = \rho^{l+1} e^{-\rho^2/2} f(\rho) \) with \( f(\rho) \) satisfying the differential equation
\[
\rho(d^2f/d\rho^2) + 2(l+1-\rho^2)(df/d\rho) - (2l + 3 - C)pf = 0.
\]
(d) Making a change of variable \( \xi = \rho^2 \), show that the differential equation for \( f(\xi) \) is
\[
\xi(d^2f/d\xi^2) + (l + 3/2 - \xi)(df/d\xi) - (1/4)(2l + 3 - C)f = 0.
\]
This is in the form of the well-known differential equation \( xF'' + (b-x)F' - aF = 0 \) whose solution is the confluent hypergeometric series
\[
F(a,b,x) = \sum_{s=0}^\infty \frac{\Gamma(a+s)\Gamma(b) x^s}{\Gamma(a)\Gamma(b+s)\Gamma(s+1)} = 1 + \frac{ax}{b} + \frac{a(a+1)x^2}{b(b+1)2!} + \ldots.
\]
It can be seen that \( F \) behaves as \( e^x \) for large \( x \). Show that the requirement that \( u_{nl} \) must be normalizable leads to \( E = (n + 3/2)\hbar\omega \) where \( n = 2s + l \). Thus each energy state \( n \) is associated with several orbital angular momentum states \( l \) according to \( l = n-2s = n, n-2, \ldots, (l \text{ or } 0) \).

2. A general angular momentum operator \( \hat{J} \) can be defined by the commutation relations of its components: \([J_x,J_y]=i\hbar J_z\). (a) Show that \([\hat{J}^2,J_z]=0\). (b) Let the common eigenvector of \( \hat{J}^2 \) and \( J_z \) be \( |\lambda, m\rangle \) such that \( \hat{J}^2 |\lambda, m\rangle = \lambda \hbar^2 |\lambda, m\rangle \) and \( J_z |\lambda, m\rangle = m \hbar |\lambda, m\rangle \). Show that \( \lambda = j(j+1) \) and, for a given \( j, m=-j,-j+1,\ldots,j \). (c) Show that possible values of \( j \) are \( 0, 1/2, 1, 3/2, \ldots \). (d) Obtain the matrices \( J_x, J_y, \) and \( J_z \) in the representation of common eigenvectors of \( \hat{J}^2 \) and \( J_z \) for \( j=3/2 \).
3. Consider a particle in a cylindrical box of radius $a$ and length $L$. Show, using cylindrical coordinates, that the possible values of the energy are $E = \frac{\hbar^2}{2m}[(n\pi/L)^2 + (\epsilon_{m\nu}/a)^2]$ while the corresponding eigenfunctions are $\psi_{m\nu}(r) = NJ_{|m|}(\epsilon_{m\nu}r/a)e^{i\nu\pi/n\pi/L}$ with $m=0,\pm1,\pm2,\ldots$, $\nu=1,2,3\ldots$, and $\epsilon_{m\nu}$ being the $\nu$th root of the Bessel function of order $|m|$. 

(Hints: $\nabla^2 = (1/r^2)\left\{\left(\partial/\partial r\right)[r^2(\partial/\partial r)] + (\partial^2/\partial \phi^2)\right\} + (\partial^2/\partial z^2)$ in the cylindrical coordinates; the Bessel function of order $n$ satisfies the differential equation $d^2J_n/dr^2 + (1/r)(dJ_n/dr) + (1 - n^2/r^2)J_n = 0$.)

4. The Green’s function $G(r)$ for a free particle is defined as the solution to the equation $\left(\hbar^2/2m(\nabla^2 + k^2)\right)G(r) = \delta(r)$. (a) Using $G(\vec{r}) = (2\pi)^{-3/2}\int G(\vec{q})e^{i\vec{q}\cdot\vec{r}}d^3\vec{q}$ and $\delta(\vec{r}) = (2\pi)^{-3}\int e^{i\vec{q}\cdot\vec{r}}d^3\vec{q}$, show that $G(\vec{q}) = (2m/\hbar^2)(k^2 - \vec{q}^2)^{-1}$. (b) Determine $G(\vec{r})$. 