The test consists of three parts.

1. The first part consists of 5 True/False questions. This part is mandatory.

2. You will be awarded credit for 3 out of 5 questions in the second part, and 3 out of 5 questions in the third part. In the following boxes, please circle the 3 that you would like us to grade.

Please do all of the following:

<table>
<thead>
<tr>
<th>Part I</th>
<th>(a)</th>
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Please circle exactly 3 of the following:

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<th>Part II</th>
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Note: Please note that a complete and correct solution will carry far more weight than several sparsely supported “solution sketches”.

FINAL SCORE (out of 175): PASS FAIL
(a) Given a fat Cantor set $F \subseteq [0,1]$, there is a function of bounded variation on $[0,1]$ which is non-differentiable precisely on $F$.

T  F

(b) If $C_k \subseteq \mathbb{R}$ is a nested sequence of non-empty, closed sets, then
\[ \bigcap C_k \neq \emptyset \]

T  F

(c) For any function $f : [0,1] \rightarrow [0,1]$, if $f' = 0$ a.e., then $f$ is a constant.

T  F

(d) Let $f_n, f$ be finite, real valued measurable functions on $[0,1]$ equipped with Lebesgue measure. Then $f_n \rightarrow f$ in $L^1[0,1]$ implies $f_n \rightarrow f$ a.e

T  F

(e) Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = x_0$ for all $x \in \mathbb{Q}$. Then $f$ is a constant.

T  F
PART II. Please choose 3 out of 5 from the following:

1. Define an $F_\sigma$-set. If $X$ is a topological space and $f : X \to \mathbb{R}$, show that the set of points at which $f$ is not continuous is an $F_\sigma$-set.

2. For a function $f : [0, 1] \to \mathbb{R}$ define what it means for $f$ to be of bounded variation. Let $f(0) = 0$ and $f(x) = x \sin(1/x)$, $x \neq 0$. Show that $f$ is not of bounded variation.

3. Define a separable metric space. Prove that every compact metric space is separable.

4. State the Baire Category Theorem. Use this to prove that $\mathbb{R}$ is uncountable.

5. Define a Hilbert space. For a Hilbert space $\mathcal{H}$ and an orthonormal basis $B$ of $\mathcal{H}$, prove that $||x||^2 = \sum_{x_j \in B} <x, x_j>, \forall x \in X$. 


PART III. Please choose 3 out of 5 from the following:

1. Suppose $A \subseteq \mathbb{R}$ is an uncountable set. Then $A$ has uncountably many accumulation points.

2. Let $f$ be a real-valued function on $[0, 1]$, and let $\lambda^*$ denote the Lebesgue outer measure. Suppose $E \subseteq [0, 1]$ and $f'$ exists and is bounded on $E$ by a constant bound $M$. Prove that $\lambda^*(f(E)) \leq M\lambda^*(E)$.

3. Suppose $\nu, \mu$ are $\sigma$-finite measures on the measurable space $(X, \mathcal{M})$, such that $\nu \ll \mu$. Prove that if $g \in L^1(\nu)$, then $g(d\nu/d\mu) \in L^1(\mu)$ and $\int gd\nu = \int g \frac{d\nu}{d\mu} d\mu$.

4. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Prove that for all $f \in (l^p)^*$ there is a fixed $t \in l^q$ such that $f(s) = \sum s_n t_n$ for all $s \in l^p$.

5. Let $\mathcal{P}$ denote the space of all polynomials on $[0, 1]$ with $L^\infty[0, 1]$ norm. If $F : \mathcal{P} \to \mathcal{P}$ is defined by

$$F\left(\sum_{k=0}^{n} a_k x^k\right) = \sum_{k=0}^{n} a_k x^{k+1},$$

show that $F$ is continuous and find $\|F\|$.