1. (50 points) This question consists of three parts.

   (a) (15 points) Let \( X_i \) be a sequence of random variables such that
   \[
   \lim_{n \to \infty} \frac{VarS_n}{n^2} = 0,
   \]
   where \( S_n = \sum_{i=1}^{n} X_i \). Show that
   \[
   \frac{S_n - ES_n}{n} \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty.\] (2)

   (b) (15 points) Now, suppose that we replace the condition (1) with the assumption that the \( X_i \) are pairwise uncorrelated and satisfy \( \sup_i EX_i^2 < \infty \). Show that the result (2) above also holds under these alternative assumptions.

   (c) (20 points) Assuming that the \( X_i \) are independent random variables such that \( \sup_i EX_i^4 < \infty \), show that the convergence in probability in (2) can be strengthened to convergence almost surely.
2. (50 points) Given a probability space \((\Omega, \mathcal{F}_0, P)\), an \(\mathcal{F}_0\)-measurable random variable \(X\) and another \(\sigma\)-field \(\mathcal{F} \subseteq \mathcal{F}_0\), the \textit{conditional expectation of} \(X\) \textit{given} \(\mathcal{F}\) is defined to be any random variable \(Y\) which is \(\mathcal{F}\)-measurable and satisfies

\[
\int_A X dP = \int_A Y dP
\]

for all \(A \in \mathcal{F}\).

(a) (15 points) To warm up, consider how this definition relates to the one taught in undergraduate probability. Specifically, suppose that \(\Omega_1, \Omega_2, \ldots\) is a finite or infinite partition of \(\Omega\) into disjoint sets each of which has positive probability (with respect to \(P\)), and let \(\mathcal{F} = \sigma(\Omega_1, \Omega_2, \ldots)\) the \(\sigma\)-field generated by these sets. Then show that on each \(\Omega_i\),

\[
E(X | \mathcal{F}) = \frac{E(X; \Omega_i)}{P(\Omega_i)}.
\]

(b) (15 points) Let \(\mathcal{F}_1 \subseteq \mathcal{F}_2\) be two \(\sigma\)-fields on \(\Omega\). Then show that

1. \(E(E(X | \mathcal{F}_1) | \mathcal{F}_2) = E(X | \mathcal{F}_1)\), and
2. \(E(E(X | \mathcal{F}_2) | \mathcal{F}_1) = E(X | \mathcal{F}_1)\).

(c) (20 points) Let \(\Omega = \{a, b, c\}\). Give an example of \(P, \mathcal{F}_1, \mathcal{F}_2,\) and \(X\) in which

\[
E(E(X | \mathcal{F}_1) | \mathcal{F}_2) \neq E(E(X | \mathcal{F}_2) | \mathcal{F}_1).
\]
3. (50 points) This problem consists of two parts.

(a) (25 points) Let $X$ be $N(0,1)$ random variable. Let

$$M(s) = E\left\{ e^{sX} \right\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left( s x - \frac{x^2}{2} \right) dx.$$ 

Show that $M(s) = e^{s^2/2}$.

(b) (25 points) Show that for any positive integer $n$ we have $E\{X^{2n+1}\} = 0$ and

$$E\{X^{2n}\} = \frac{(2n)!}{2^n n!} = (2n-1)(2n-3)\ldots3\cdot1.$$ 

(Hint: Note that $e^{s^2/2} = \sum_{k=0}^{\infty} \frac{s^{2k}}{2^k k!}$)
4. (50 points) This question consists of two parts.

(a) (25 points) Show that for any c.d.f. $F$ and any $a \geq 0$

$$\int [F(x + a) - F(x)]dx = a$$

(b) (25 points) Let $X$ be a random variable with range $\{0,1,2,\ldots\}$. Show that if $EX < \infty$ then

$$EX = \sum_{i=1}^{\infty} P(X \geq i)$$
5. (50 points) Let $U_1, \ldots, U_n$ be i.i.d. random variables having uniform distribution on $[0,1]$ and $Y_n = (\prod_{i=1}^{n} U_i)^{-1/n}$. Show that

$$\sqrt{n}(Y_n - e) \xrightarrow{D} N(0, e^2)$$

where $e = \exp(1)$. 
6. (50 points) Let \( \phi \) be a UMP test of level \( \alpha \in (0, 1) \) for testing simple hypothesis \( P_0 \) vs \( P_1 \). If \( \beta \) is the power of the test, show that \( \beta \geq \alpha \) with equality if and only if \( P_0 = P_1 \).
7. (50 points) Let $X_1, \ldots, X_n$ be independently and identically distributed with density

$$f(x, \theta) = \frac{1}{\sigma} \exp \left\{ - \frac{x - \mu}{\sigma} \right\}, \ x \geq \mu,$$

where $\theta = (\mu, \sigma^2), -\infty < \mu < \infty, \sigma^2 > 0$.

(a) Find maximum likelihood estimates of $\mu$ and $\sigma^2$.

(b) Find the maximum likelihood estimate of $P_\theta(X_1 \geq t)$ for $t > \mu$. 
8. (50 points) Let \( X_1, \ldots, X_n \) be iid from Bernoulli distribution with unknown probability of success \( P(X_1 = 1) = p \in (0, 1) \).

(a) (20 points) Show that \( S = \sum_{i=1}^{n} X_i \) is a complete and sufficient statistic.

\[
\text{(Hint: } \sum_{k=0}^{n} \binom{n}{k} g(k) p^k (1 - p)^{n-k} = (1 - p)^n \left( \sum_{k=0}^{n} \binom{n}{k} g(k) \zeta^k \right) \text{ where } \zeta = \frac{p}{1 - p} \text{)}
\]

(b) (30 points) Find UMVUE for \( p^m \) when \( m \leq n \) is a positive integer.