Mathematical Modeling Qualifier Exam
Oct, 2007

Name ______________________________ **ID** ____________________

There are two parts in this Exam.

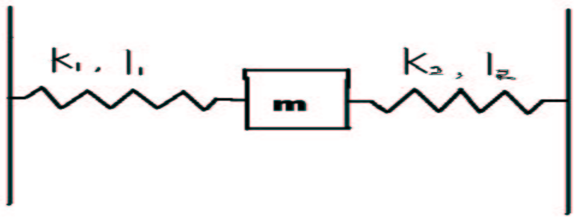
Do any 3 problems out of the 4 problems in Part I, and do any 3 problems out of the 4 problems in Part II.

You must show all work to receive full credit. If you do more problems than what are required, you MUST indicate what problems you want to be graded. Otherwise, they may be randomly chosen to grade.

Part I:

1. Suppose that a mass m were attached to two walls distance d apart. The spring coefficients and natural lengths of the springs are k_1, l_1 and k_2, l_2, respectively.

Figure 1:



(a) Find the equation describing the movement of the mass.

(b) Find the equilibrium position of the mass.

(c) Solve the equation.

2. The Van der Pol oscillator is described by the following nonlinear differential equation:

$$\frac{d^2x}{dt^2} - \epsilon\frac{dx}{dt}(1-x^2) + \omega^2 x = 0,$$

where $\epsilon \geq 0$.

(a) Find the equilibrium position and determine its stability.

(b) If displacements are large, what do you expect happens? Give details of analysis.

(c) Sketch the trajectories in the phase plane if $\omega = 1$ and $\epsilon = \frac{1}{10}$. Describe any interesting features of the solution.

3. Consider the traffic model

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = 0,$$

with the initial condition

$$\rho(x,0) = \begin{cases} 1 & x \leq -2 \\ 3 & -2 < x < 0 \\ 0 & 0 < x < 4 \\ 1 & x \geq 4. \end{cases}$$

Suppose that the traffic flux is given by the function $q(\rho) = 2\rho(3-\rho)$. Sketch the characteristics and the shocks. Find formulas for the density $\rho(x,t)$ and the shock $x_s(t)$.

4. The traffic flow in a highway with entrances and exits may be modeled as

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = \beta,$$

where

$$\beta(x,t) = \begin{cases} 0 & x \leq 0 \\ \beta_0 & 0 < x < 2 \\ 0 & x \geq 2. \end{cases}$$

Suppose that $\beta_0 = 2$, $q(\rho) = 4\rho(2-\rho)$ and the initial density is $\rho(x,0) = 0$.

(a) Sketch the graphs of characteristics and typical densities for different time t.

(b) Find formulas for the density $\rho(x,t)$ and the characteristics.

Part II:

1. Cosider the specific Holling-Tanner model

$$x' = x\left(1 - \frac{x}{7}\right) - \frac{6xy}{(7+7x)}, \quad y' = 0.2y\left(1 - \frac{Ny}{x}\right).$$

   where N is a constant with $x(t) \neq 0$ and $y(t)$ representing the populations of prey and predators, respectively. Sketch phase portraits when (a) $N = 2.5$, and (b) $N = 0.5$.

   Hint: Construct a phase plane diagram in the usual way. Find the critical points, linearize around each one, determine the isoclines, and plot a phase plane portrait.

2. Show that the system

$$x' = -y + x(1 - 2x^2 - 3y^2), \quad y' = x + y(1 - 2x^2 - 3y^2).$$

   has a unique limit cycle.

3. Consider the Hamiltonian system

$$\dot{x} = y,$$
$$\dot{y} = x - x^3$$

   (a) Find all critical points and determine their local stability

   (b) Find the Hamiltonian of the system and sketch a phase portrait.

4. Consider the Hénon map given by

$$x_{n+1} = 1 - \alpha x_n^2 + y_n,$$
$$y_{n+1} = \beta x_n,$$

   where $\alpha > 0$ and $|\beta| < 1$.

   (a) Find period-one critical points of the system, and describe how you determine their local stability.

   (b) Show that when $\alpha = \frac{3(\beta-1)^2}{4}$ for fixed β, two period-two critical points bifurcate from a period-one critical point.