This examination consists of two parts, A and B. Part A contains six problems of which you must select four to do. Part B contains three problems of which you must select two to do. Each problem in part A is worth 15 points and each problem in part B is worth 20 points. Only hand-in your solutions to four problems from part A and two from part B. Please do not turn-in more solutions since only the first four solutions from part A will be graded and only the first two solutions from part B will be graded.

Begin each problem on a new sheet of paper and be sure to label each page of your work with the problem number and your name.

In each question, if you appeal to a theorem within your solution, you must carefully state the entire theorem. All graphs, unless otherwise stated, should be understood to be finite and simple.
**Problem A1:** A tournament is a complete graph in which every edge has been given an orientation. Prove that every tournament has a directed Hamiltonian path.

**Problem A2:** Solve the recurrence

\[ a_n = 5a_{n-1} - 6a_{n-2} \quad (\text{for } n \geq 2), \]

with initial conditions \( a_0 = 1 \) and \( a_1 = 1 \).

**Problem A3:** Suppose that \( G \) is a connected planar graph that can be drawn in the plane so that all faces have an even number of edges on their boundary. Prove that the vertices of \( G \) can be properly 2-colored.

**Problem A4:** All points of the plane that have integer coordinates are colored so that each such point receives one of the three colors: red, blue or green. Prove that there must be a rectangle whose four corner vertices are all of the same color.

**Problem A5:** Prove that if every chain and every antichain of a poset \( P \) is finite, then \( P \) is finite.

**Problem A6:** Let \( G \) be a graph in which any two odd cycles intersect.

a) Prove that \( G \) is 5-colorable.

b) Give an example to show that 4 colors do not suffice.
Problem B1:
   a) Find, with a proof, the number of edges in the extremal graph on 6 vertices without $K_4$ as a subgraph.
   b) Find, with a proof, the number of edges in the extremal graph on 6 vertices without $C_4$ as a subgraph.

Problem B2: Prove the given identity:
   a) 
   \[
   \sum_{i=0}^{n} \binom{a}{i} \binom{b}{n-i} = \binom{a+b}{n}
   \]
   b) 
   \[
   \sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}.
   \]

Problem B3: Prove or disprove: If $G$ is a connected, simple graph that does not contain $P_4$ or $C_3$ as an induced subgraph, then $G$ is a complete bipartite graph.