This examination consists of two parts, A and B. Part A contains six problems of which you must select four to do. Part B contains three problems of which you must select two to do. Each problem in part A is worth 15 points and each problem in part B is worth 20 points. Only hand-in your solutions to four problems from part A and two from part B. Please do not turn-in more solutions since only the first four solutions from part A will be graded and only the first two solutions from part B will be graded.

Begin each problem on a new sheet of paper and be sure to label each page of your work with the problem number and your name.

In each question, if you appeal to a theorem within your solution, you must carefully state the entire theorem. All graphs, unless otherwise stated, should be understood to be finite and simple.
PART A: (15 points each) Do any four. 

Problem A1:

- **a)** Show that if \( G \) is a 2-connected graph containing a vertex that is adjacent to at least three vertices of degree 2, then \( G \) is not Hamiltonian.
- **b)** The subdivision graph \( S(G) \) of a graph \( G \) is the graph obtained from \( G \) by replacing each edge \( uv \) by a vertex \( w \) and edges \( uw \) and \( vw \). Determine, with a proof, all graphs \( G \) for which \( S(G) \) is Hamiltonian.

Problem A2: Recall that a set \( S \) of vertices of a graph \( G \) is independent if every two vertices of \( S \) are not adjacent in \( G \). The independence number, \( \beta(G) \), of a graph \( G \) is the maximum cardinality among independent sets of vertices of \( G \). Prove that a graph \( G \) is bipartite if and only if \( \beta(H) \geq \frac{|V(H)|}{2} \), for every subgraph \( H \) of \( G \).

Problem A3: A simple planar graph \( G(V, E) \) has only vertices of degree 3, 4, 5, and 6, with the same number of each type. Find the order \( |V| \) and the size \( |E| \) of \( G \).

Problem A4: All points of the plane that have integer coordinates are colored so that each such point receives one of the three colors: red, blue or green. Prove that there must be a rectangle whose four corner vertices are all of the same color.

Problem A5: Solve the recurrence relation

\[ a_0 = 1 \quad \text{and} \quad a_n = 3 \sum_{i=0}^{n-1} a_i, \quad \text{for all } n \geq 1. \]

Problem A6: A function \( f : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \) is monotone if \( x < y \) implies \( f(x) \leq f(y) \). Determine the number of such monotone functions.
PART B: (20 points each) Do any two. Time: 1 hour and 20 minutes.

PROBLEM B1:
   a) Find, with a proof, the number of edges in the extremal graph on 6 vertices without $K_4$ as a subgraph.
   b) Find, with a proof, the number of edges in the extremal graph on 6 vertices without $C_4$ as a subgraph.

PROBLEM B2: Consider the poset $(P, \leq)$ where $P$ is the set of all subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ with odd cardinality and $\leq$ is the inclusion relation.
   a) Find the number of elements in $P$.
   b) Find all minimal and maximal elements of $(P, \leq)$.
   c) Determine the length $\ell$ and the width $w$ of $(P, \leq)$.
   d) Give an example of a chain of cardinality $\ell$ and an antichain of cardinality $w$.

PROBLEM B3: The automorphism group $\text{Aut}(G)$ of the graph $G$ shown below consists of four permutations.
   a) List all elements of $\text{Aut}(G)$ using cycle notation.
   b) Find the cycle index of the group $\text{Aut}(G)$.
   c) Find the number of different 3-colorings of the vertices of $G$.
   d) Find the number of different labelings $\ell(G)$ of the vertices of the graph $G$ if the available labels are $a, b, c, d, e, f$.

\[ G: \begin{array}{c}
1 & 2 \\
3 & 4 \\
5 & 6
\end{array} \]