Ph.D. Qualifying Examination in Algebra

Department of Mathematics
University of Louisville

17 th October 2003

The questions on this examination are presented in three sections - one section on **Groups***, one on* **Rings** *and one on* **Fields***. You should attempt at least two questions from each section. If you attempt three questions from one section you will be awarded the scores on your best two questions from that section.*

1 Groups

Do any two questions from this section.

1. Let G be an abelian group. For any integer $n > 0$ show that the map $\phi : x \mapsto x^n$ is a homomorphism of G into G. Characterize the kernel of ϕ. Show that if G is finite and n is relatively prime to the order of G then ϕ is an isomorphism. Deduce that in this case, for each element $g \in G$ there is a unique element $x \in G$ such that $g = x^n$.

2. For any group G the *center* $Z(G)$ of G is the set of elements of G that commute with all elements of G,

$$Z(G) = \{x \in G \mid xg = gx \ \forall \ g \in G\}.$$

   (a) Prove that if $G/Z(G)$ is cyclic then G is abelian.

   (b) Suppose that $G/Z(G)$ is abelian. Prove or disprove that G must also be abelian.

3. The Sylow theorems may be summarized as follows:

   If p^n is the largest power of a prime p dividing the order of a finite group G then G has a subgroup of order p^n. Any two such subgroups are conjugate in G and the number of such subgroups is conjugate to 1 *(mod p) and divides their common index.*

   (a) Let p and q be primes with $p > q$ and let G be a group of order pq. Then G has a $p-$Sylow subgroup P of order p and a $q-$Sylow subgroup Q of order q. Prove that $G = PQ$.

   (b) Show that every group of order 56 has a proper normal subgroup.

2 Rings

Do any two questions from this section.

1. (a) Let R be the ring given by defining new operations on the integers $\mathbf{Z}$ by letting $m \oplus n = m + n - 1$ and $m \otimes n = m + n - m.n$ where $+$ and . are the usual operations of addition and multiplication on $\mathbf{Z}$. Define $\phi : \mathbf{Z} \rightarrow R$ by $\phi(n) = 1 - n$. Prove that ϕ is an isomorphism.

   (b) Note that ϕ maps the identity element 1 of $\mathbf{Z}(+,.)$ to the identity element 0 of $R(\oplus, \otimes)$. Prove that if θ is a homomorphism of a commutative ring with identity S to an integral doman D then θ maps the identity element of S to the identity element of D.

   (c) Show that a homomorphism of a ring V with identity into a ring W with identity does not necessarily map the identity element of V to the identity element of W.

2. Let R be a commutative ring with ideals I, J such that $I \subseteq J \subseteq R$. It is well known that J/I is an ideal of R/I.

   (a) Show that the factor ring $(R/I)/(J/I)$ is isomorphic to R/J. *Hint: Define a homomorphism from R/I onto R/J and apply the fundamental homomorphism theorem for rings.*

   (b) Using the result of Part (a), or otherwise, determine all prime ideals and all maximal ideals of $\mathbf{Z_n}$.

3. Let R be a ring with identity and let $x \in R$ have a right inverse y and a left inverse z. Prove that $y = z$.

   Suppose that K is a ring with identity 1 and let x be an element of K that has a right inverse y but has no left inverse. Prove that the function $\phi : z \mapsto y + zx - 1$ maps the set of right inverses of x into itself. Show that ϕ is one-to-one but not onto and deduce that if an element of a ring has more than one right inverse then it has an infinite number of right inverses.

3 Fields

Do any two questions from this section.

1. Let $K \subseteq E \subseteq F$ where E is a finite extension field of the field K and F is a finite extension field of the field E. It is well known that $[F : K] = [F : E][E : K]$.

   Using this result, or otherwise, prove that $[\mathbf{Q}(\sqrt{2} + \sqrt{3}) : \mathbf{Q}] = 4$.

2. Construct a splitting field for the polynomial $x^3 - 2$ over $\mathbf{Q}$.

3. Define $\phi : GF(2^2) \to GF(2^2)$ by $\phi(x) = x^2$, for all $x \in GF(2^2)$.

   (a) Show that ϕ is an isomorphism.

   (b) Choose an irreducible polynomial $p(x)$ to represent $GF(2^2)$ as $\mathbf{Z_2}[x]/ < p(x) >$. For your choice of $p(x)$ give an explicit computation of $\phi(\beta)$ for each element β of $\mathbf{Z_2}[x]/ < p(x) >$.