

COMBINATORICS QUALIFYING EXAM
October, 2003

This examination consists of two parts, A and B. Part A contains **five** problems of which you must select four to do. Part B contains **three** problems of which you must select two to do. Each problem in part A is worth 15 points and each problem in part B is worth 20 points. Only hand-in your solutions to four problems from part A and two from part B. Begin each problem on a new sheet of paper and be sure to label each page of your work with the problem number and your name. You have two hours to complete part A and one hour and 20 minutes to complete part B. There will be ten-minute break between parts A and B.

In each question, if you appeal to a theorem within your solution, you must carefully state that theorem. All graphs, unless otherwise stated, should be understood to be finite and simple.

PART A: (15 points each) *Do any four.*

Time: 2 hours.

PROBLEM A1: Let G be a graph in which any two odd cycles intersect.

- a) Prove that G is 5-colorable.
- b) Give an example to show that 4 colors do not suffice.

PROBLEM A2: Find all 3-regular plane graphs in which all faces are triangles. Prove your list is complete.

PROBLEM A3: Let

$$S(n) = \{(A, B) : \emptyset \subseteq A \subseteq B \subseteq \{1, 2, \dots, n\}\}.$$

- a) Find a recurrence relation for $a_n = |S(n)|$.
- b) Find a compact formula for $|S(n)|$. Justify your answer.

PROBLEM A4: Prove that if every chain and every antichain of a poset P is finite, then P is finite.

PROBLEM A5: Consider the ways to distribute n identical balls to five different boxes with the first four boxes receiving between 3 and 8 balls.

- a) Write the ordinary generating function for the number of these distributions.
- b) In how many ways can 25 balls be distributed in this way?

PART B: (20 points each) *Do any two.*

Time: 1 hour and 20 minutes.

PROBLEM B1: Prove the given identity:

a)

$$\sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1} - 1}{n+1}$$

b)

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$

PROBLEM B2:

- a) State the definition of what it means for a graph to be an interval graph.
- b) State the definition of what it means for a graph to be a perfect graph.
- c) Prove directly that every interval graph is perfect.

PROBLEM B3: Use inclusion-exclusion to find a formula for the number of 1-factors in the graph obtained from $K_{n,n}$ by removing the edges of a perfect matching.