Ph.D. Qualifying Examination in Algebra

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1 Groups

Do any three problems out of the set below. Specify clearly which problems are you attempting.

1. Let G be a simple group of order 168. How many elements are there in G of order 7? What can you say about the number of elements of order 7 in G if G is not simple?

2. Show that a group with precisely two conjugacy classes is simple. Find an example of such a group.

3. Determine up to isomorphism all the abelian groups of order 108

4. Prove that if G is a finite group and H is a maximal normal subgroup then the quotient group G/H is simple. Prove that every finite group has a composition series.

5. Let G be a group of odd order, and let p be an odd prime that divides $|G|$. Prove that there are even number of elements of G of order p.

Do any four problems out of the sets **Rings** *and* **Fields** *below; do at least one problem in each set. Specify clearly which problems are you attempting.*

2 Rings

1. Consider the ideals $(3), (1 + i), (2)$ in the ring $\mathbb{Z}[i]$ of the Gaussian integers.

   (a) Are these ideals maximal?

   (b) Are these ideals prime?

(c) Which of the quotient rings are fields?

(d) What are the characteristics of the fields above in 1c?

2. Let F be a field, and let S be the ring of all 2×2 matrices over F. Prove or disprove each statement:

   (a) The center $Z(S)$ of S is $\{\lambda I : \lambda \in F\}$, where I is the identity matrix.

   (b) $Z(S)$ is an ideal of S.

   (c) Let $A, B \in S$, then $(AB - BA)^2$ is in $Z(S)$.

3. (a) Prove that if D is a unique factorization domain, then so is the polynomial ring $D[x]$.

   (b) Prove or disprove that $\mathbb{Z}[x]$ is a unique factorization domain.

   (c) Decide if the following equations contradict the above statements:

$$x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1)$$

$$12x^3 + 4x^2 = 4x^2(3x + 1) = 2x(6x^2 + 2x)$$

3 Fields

1. Let K be a field and $f \in K[x]$ be an irreducible separable polynomial of degree 2. Determine the Galois group G of f.

2. Find the splitting field F of $f = x^4 + x^2 + 1$ over $\mathbb{Q}$ and calculate $[F : \mathbb{Q}]$.

3. Given $g = x^2 + x + 1 \in \mathbb{Z}_2[x]$, construct a field of four elements. Construct the addition and multiplication tables for this field.