Combinatorics Qualifying Examination

May 28, 2003

This examination consists of two parts, A and B. Part A consists of five problems and Part B consists of three problems. You are to do any four problems from Part A and any two problems from Part B. Each problem from Part A is valued at 15 points, and each problem in Part B is worth 20 points. Only hand in four problems from Part A and two problems from Part B. Begin each problem on a new sheet of paper, and only write on one side of the paper. You have two hours to complete Part A of the exam. When you are ready to hand in your exam, assemble the problems in numerical order, write your name on the front page, and initial the other pages. There will be a ten minute break before Part B. You have one hour and 20 minutes to complete Part B of the exam.

Part A (15 points each) Do any <u>four</u>.

A1. Let G be a connected simple graph whose line graph $L(G)$ is cubic.

(a) Prove that for every edge $e = uv$ of G, $\deg_G u + \deg_G v = 5$.

(b) Prove that the graph G is bipartite.

A2. The *Cartesian product* of graphs G and H, written $G \times H$, is the graph with vertex set $V(G) \times V(H)$, and two vertices (u, v) and (x, y) of $G \times H$ are adjacent if and only if either (1) $u = x$ and $vy \in E(H)$, or (2) $v = y$ and $ux \in E(G)$.

The *n-cube* Q_n is defined by $Q_1 = K_2$ and $Q_n = Q_{n-1} \times K_2$ for $n \geq 2$. Let G be a graph with chromatic number $\chi(G) = k \geq 2$.

(a) Let C_5 denote the 5-cycle. Draw a plane embedding of the graph $C_5 \times K_2$.

(b) Prove that $\chi(G \times K_2) = k$.

(c) Prove that $\chi(G \times Q_n) = k$ for each $n \geq 1$.

A3. A graph is *hamiltonian* if it contains a spanning cycle (a cycle through every vertex). A *hamiltonian path* is a spanning path (a path through every vertex). Prove or disprove the following:

(a) Every cubic hamiltonian graph has edge-chromatic number 3.

(b) There exists a cubic eulerian graph with edge-chromatic number 3.

(c) Every cubic graph possessing a hamiltonian path has edge-chromatic number 3.

A4. Let $X = \{a, b, c\}$. Find the number $N(n)$ of words (sequences) of length n in which the letters are taken from X and the letter a appears an even number of times. Use two different counting techniques:

(a) Exponential generating functions.

(b) Justify that $N(n)$ satisfies the following recurrence relation: $N(n+1) = N(n) + 3^n$. Prove the compact formula for $N(n)$ by induction.

A5. The *crossing number* of a graph G is the minimum number of crossings in a drawing of G in the plane. Let G be the complete bipartite graph $K_{4,3}$.

(a) Prove that G is not a planar graph.

(b) Prove that the crossing number of G is not 1.

Hint: Suppose there is a drawing of G in the plane with one crossing v. Consider the new (plane) graph H with one extra vertex v. Use Euler's formula to find the number of regions of H. What are the degrees of the faces of H? Obtain a contradiction.

(c) Prove that the crossing number of G is at most 2.

Part B (20 points each) Do any <u>two</u>.

B1. If G is a simple graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$, then its *adjacency matrix* is the $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = 1 \text{ if } v_i v_j \text{is an edge of } G, \text{ and } a_{ij} = 0 \text{ otherwise.}$$

Let G be a simple $(5, q)$ graph, with an adjacency matrix A. Suppose that

$$A^2 = \begin{bmatrix} 2 & 1 & 1 & 2 & 2 \\ 1 & 4 & 3 & 2 & 2 \\ 1 & 3 & 4 & 2 & 2 \\ 2 & 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 2 & 3 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 2 & 7 & 7 & 4 & 4 \\ 7 & 8 & 9 & 9 & 9 \\ 7 & 9 & 8 & 9 & 9 \\ 4 & 9 & 9 & 6 & 7 \\ 4 & 9 & 9 & 7 & 6 \end{bmatrix}.$$

(a) How many (non-identical) 3-cycles does G contain?

(b) Determine q, the number of edges of G.

(c) Determine diam G, the diameter of G.

(d) Determine rad G, the radius of G.

(e) Draw the graph of G.

B2. Consider the poset $(P, \leq)$, where

$$P = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 36, 48, 54, 72\}$$

and for $a, b \in P$, $a \leq b$ if and only if a divides b.

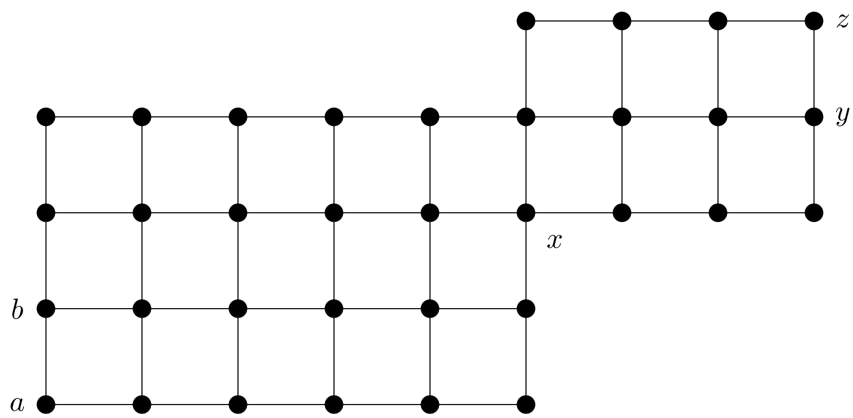
(a) Represent P graphically by its Hasse disgram.

(b) Find all maximal and all minimal elements of $(P, \leq)$. Give an example of a maximum chain and an example of a maximum antichain in $(P, \leq)$.

(c) State Dilworth's theorem and illustrate it using the above poset P as an example.

(d) The cardinality of poset P is 16. Show that every poset of cardinality 16 must contain either a chain of cardinality 6 or an antichain of cardinality 4.

B3. Consider the graph G with 34 vertices and 54 edges presented below.



(a) What is the maximum number of pairwise vertex internally-disjoint b, y-paths in G?

(b) What is the maximum number of pairwise edge-disjoint b, y-paths in G?

(c) Find the number of shortest (not necessarily disjoint) a, x-paths in G.

(d) Find the number of shortest (not necessarily disjoint) a, z-paths in G passing through x.

(e) Find the number of shortest (not necessarily disjoint) a, z-paths in G.

(f) What is the length of a longest a, z-path in G?

Justify all your answers!