Analysis Qualifier Exam

October 13, 2004

Following are ten problems divided into two groups of five. Do five problems, including at most three from each group.

Group 1

Problem 1. Find a set $S \subset \mathbb{R}$ of Lebesgue measure zero that is uncountable in every interval.

Problem 2. If f is Lebesgue integrable on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f(x)\cos(nx)\,dx = 0.$$

Problem 3. Suppose that λ is Lebesgue measure and μ is counting measure on $I = [0,1]$. (Regard both as Borel measures.) If $\Delta = \{(x,x) : x \in I\}$ and f is the characteristic function of Δ, then calculate

$$\int_I \left(\int_I f\,d\lambda \right) d\mu, \quad \int_I \left(\int_I f\,d\mu \right) d\lambda \text{ and } \int_{I\times I} f\,d\mu \times d\lambda.$$

Reconcile your results with Fubini's theorem.

Problem 4. Suppose $f \in L^1([0,1])$ and

$$\left| \int_a^b f \right| \le b - a$$

whenever $0 \le a < b \le 1$. Prove $|f(x)| \le 1$ a.e.

Problem 5. Let

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$$

(a) Show that the Lebesgue integral $\int_0^\infty f$ does not exist.
(b) Show that the improper Riemann integral $\int_0^\infty f(x)\,dx$ does exist.

Group 2

Problem 6. If f is monotone on $[0,1]$, then there is a sequence of continuous functions $f_n \to f$.

Problem 7. If $f : [a,b] \to \mathbb{R}$ is continuous, then

$$G = \{(x, f(x)) : a \leq x \leq b\} \subset \mathbb{R}^2$$

has Lebesgue measure 0.

Problem 8. Prove that a dense G_δ subset of $\mathbb{R}$ is uncountable.

Problem 9. If $f \in L^\infty([0,1])$, then $f \in L^p([0,1])$ for all $p \geq 1$ and

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p.$$

Problem 10. If $f_n : [a,b] \to \mathbb{R}$ is a sequence of continuous functions that converges uniformly, then

$$\mathcal{F} = \{f_n : n = 1, 2, 3, \cdots\}$$

is equicontinuous.