Combinatorics Qualifying Exam
October, 2004

This examination consists of two parts, A and B. Part A contains **five** problems of which you must select four to do. Part B contains **three** problems of which you must select two to do. Each problem in part A is worth 15 points and each problem in part B is worth 20 points. Only hand-in your solutions to four problems from part A and two from part B. Begin each problem on a new sheet of paper and be sure to label each page of your work with the problem number and your name. You have two hours to complete part A and one hour and 20 minutes to complete part B. There will be ten-minute break between parts A and B.

In each question, if you appeal to a theorem within your solution, you must carefully state that theorem. All graphs, unless otherwise stated, should be understood to be finite and simple.

PART A: **(15 points each)** *Do any four.* **Time: 2 hours.**

PROBLEM A1: A graph G is a *chordal graph* if every cycle C of G contains an edge joining two non-consecutive vertices of C. A graph $G(V, E)$ is an *interval graph* if there is an assigment f that assigns an interval I_v of the real line to each vertex $v \in V(G)$ such that $uv \in E$ if and only if $I_u \cap I_v \neq \emptyset$. Prove that every interval graph is a chordal graph.

PROBLEM A2: A *tournament* is an complete graph in which every edges has been given an orientation. Prove that every tournament has a directed Hamiltonian path.

PROBLEM A3: Solve the recurrence

$$a_n = 5a_{n-1} - 6a_{n-2} \qquad (\text{ for } n \geq 2),$$

with initial conditions $a_0 = 1$ and $a_1 = 1$.

PROBLEM A4: Suppose that G is a connected planar graph that can be drawn in the plane so that all faces have an even number edges on their boundary. Prove that the vertices of G can be properly 2-colored.

PROBLEM A5: Let h_n denote the number of nonnegative integral solutions of the equation:

$$x_1 + x_2 + x_3 + x_4 = n.$$

a) Write the ordinary generating function for h_n.
b) What is h_{25}?

PART B: **(20 points each)** *Do any two.* **Time: 1 hour and 20 minutes.**

PROBLEM B1: Prove the given identity:

a)

$$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}$$

b)

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

PROBLEM B2:

a) State the definition of what it means for a graph to be a perfect graph.

b) A graph $G(V, E)$ is a *comparability graph* if there is a partial order P on V so that $uv \in E$ if and only if u and v are comparable in P. Prove that every comparability graph is perfect.

PROBLEM B3: Use inclusion-exclusion to find a formula for the number of 1-factors in the graph obtained from $K_{n,n}$ by removing the edges of a perfect matching.