Ph.D. Qualifying Examination in Algebra
Department of Mathematics
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1 Groups

Do any two questions from this section.

1. Let G be a finite group of order $4q^2$ where $q \geq 5$ is a prime number. Prove that there is a normal subgroup of order q^2.

2. Prove that an abelian group has a composition series if and only if it is finite.

3. Determine up to isomorphism all the abelian groups of order 1500.

2 Rings

Do any two questions from this section.

4. Let $Z[\sqrt{-3}] = \{m + n\sqrt{-3} : m, n \in Z\}$. Prove that $Z[\sqrt{-3}]$ is not a unique factorization domain. (Hint: You may want to use the function $N(a) = m^2 + 3n^2$ for all $a = m + n\sqrt{-3} \in Z[\sqrt{-3}]$.)

5. Recall that an element a from a ring R is nilpotent if $a^n = 0$ for some positive integer n. Let R be a ring with no nonzero nilpotent elements. Prove that if $f(x) \in R[x]$ is a zero divisor, then there exists $b \in R$ such that $b \cdot f(x) = 0$.

6. Prove the following result. In a commutative ring R with identity $1_R \neq 0$ an ideal P is prime if and only if R/P is an integral domain.

3 Fields

Do any two questions from this section.

7. Let Q be the field of rational numbers. In the field of complex numbers C, prove that the subfields $Q(i)$ and $Q(\sqrt{2})$ are isomorphic as vector spaces over Q, but not as fields.

8. Let $f(x) = x^5 - 6x + 2$
(a) Show that $f(x)$ is irreducible over Q, and that in C, it has exactly three real roots. (For the last part you need calculus.)
(ii) Deduce that if L is the splitting field of f over Q, $G = Gal(L/Q)$, when identified with a subgroup of S_5, contains a 5-cycle and a 2-cycle. (This implies that $G = S_5$, but you don't need to prove this.)

9. Prove or disprove the following:
(i) There are precisely p automorphisms of the field of order p.
(ii) All fields with 49 elements are isomorphic.
(iii) $Q(\sqrt{2}, \sqrt{3})$ is a simple extension of Q.