1 Basic Analysis.

Do **three** problems from this section. Clearly state which three problems you would like graded.

1. 1a. State some version of the Baire Category Theorem.

   1b. Prove that the set of irrational numbers is not the countable union of closed subsets of $\mathbb{R}$.

2. Suppose that (X, d) is a separable metric space. Show that every uncountable subset of X has a limit-point in X.

3. 3a. State a condition which is necessary and sufficient for a function $f : [0, 1] \to \mathbb{R}$ to be Riemann integrable.

   3b. Give an example of a characteristic function of a closed set which is not Riemann integrable. Explain why your example works.

4. Consider the power series:

$$f(x) = \sum_{n=0}^{\infty} \frac{n^2}{3^n} x^n.$$

   Show that f is continuous and differentiable on $(-3, 3)$.

2 Measure and Integration.

Do **three** problems from this section. Clearly state which three problems you would like graded.

5. 5a. State what it means for a function $f : [0, 1] \to \mathbb{R}$ to be absolutely continuous.

   5b. Give an example of a continuous nondecreasing function $f : [0, 1] \to [0, 1]$ such that f is not absolutely continuous. Explain why your function is not absolutely continuous.

6. Suppose that $M \subset [0, 1]$ is such that $M \cap P \neq \emptyset$ and $M^c \cap P \neq \emptyset$ for every closed set $P \subseteq [0, 1]$ of positive measure. Show that M is non-measurable.

7. Suppose that $f \in L^1([0, 1])$ and $\{g_n\}$ is a sequence of bounded measurable functions defined on $[0, 1]$ which converges uniformly to g. Show that

$$\lim_{n\to\infty} \int_{[0,1]} f g_n = \int_{[0,1]} f g.$$

8. Show that the smallest σ-algebra containing $\mathcal{G} = \{[a, b) : a, b \in \mathbb{R}\}$ is the set of Borel sets.

9. Suppose $f, g \in L^1((-\infty, \infty))$. Define $h(x) = \int_{-\infty}^{\infty} f(t)g(x - t)dt$ for all $x \in \mathbb{R}$. Show that and $\|h\|_1 = \|f\|_1 \|g\|_1$ and conclude that h is finite a.e.

3 Functions Spaces.

Do **two** problems from this section. Clearly state which two problems you would like graded.

10. Define $T : C([0,1]) \to \mathbb{R}$ by $T(f) = \sum_{n=0}^{\infty} \frac{1}{2^n} f(\frac{1}{2^n})$.

    10a. Explain why $T(f)$ is finite.

    10b. Show that T is a bounded linear operator on $C([0,1])$.

    10c. Find a BV function $g : [0,1] \to \mathbb{R}$ such that $T(f) = \int_0^1 f dg$ for all $f \in C([0,1])$.

11. Suppose that $\{f_n\}$ is a sequence of differentiable functions defined on $[0,1]$ such that $f_n(0) = 0$ for all $n \in \mathbb{N}$ and $|f_n'(x)| \leq 1$ for all $x \in [0,1]$ and for all $n \in \mathbb{N}$. Show that $\{f_n\}$ has a subsequence which converges uniformly to some function f which is Lipschitz.

12. Find an uncountable subset $\mathcal{U}$ of $L^\infty([0,1])$ such that $\|f - g\|_\infty = 2$ for all $f, g \in \mathcal{U}$ with $f \neq g$.