Preliminary Exam
COMBINATORICS
May, 2004

This examination consists of two parts, A and B. Each parts contains five problems. Each problem in part A is worth 15 points and each problem in part B is worth 10 points. You have to solve any four problems out of part A and any four problems out of part B. Begin each problem on a new sheet of paper, and only write on one side of the paper. Only hand in those selected eight problems. You have 3 hours and 30 minutes to complete the exam.

PART A (15 points each) Do any <u>four</u>.

PROBLEM **A1**.

Suppose that G is a cubic graph.

a) Prove that if G is bridgeless, then G has a perfect matching.
b) Find a cubic graph with no perfect matching.

PROBLEM **A2**.

Determine the number of functions from $A = \{1, \ldots, n\}$ onto $B = \{1, 2, 3, 4\}$.

PROBLEM **A3**.

There are n lines in the plane in general position (that is no two lines are parallel and no three lines have a common point). Use Euler's formula to find the number of regions determined by the lines.

Hint: Draw a circle large enough to contain all the intersection points inside it.

PROBLEM **A4**.

Use Dilworth's theorem to prove that every sequence of $n^2 + 1$ distinct numbers contains a monotone subsequence of length at least $n + 1$.

PROBLEM A5.

A graph is a *comparability graph* if there is a partial ordering of its vertex set such that two vertices are adjacent if and only if they are comparable. Show that every comparability graph is perfect.

PART B (10 points each) Do any four.

PROBLEM B1.

Suppose that G is a finite simple graph that contains three edges whose removal destroys all cycles in G. Prove that G is a planar graph.

• PROBLEM B2.

Define $g: N \rightarrow Z$ recursively by

$$g(0) = 4, \ g(1) = 5, \ g(2) = 19,$$

$$g(n) = 2g(n\text{-}1) + g(n\text{-}2) - 2g(n\text{-}3), \text{ for all } n \geq 3.$$

Find the explicit formula for $g(n)$.

• PROBLEM B3.

Prove that a k-connected graph with at least $2k$ vertices contains a matching with k edges.

• PROBLEM B4.

Let w_n be the number of n-letter words that use letters from the alphabet $\{A, B\}$ and have no two consecutive A's. Find the recurrence for w_n.

PROBLEM B5.

Imagine an $n \times k$ chessboard and a chess piece which is allowed to move only one step forward or backward or one step to either side(no diagonal moves). Assume that $n \geq 2$ and $k \geq 2$. Find all the values of n and k for which this $n \times k$ chessboard has a hamiltonian tour and justify your answer.