Mathematical Modeling Qualifier Exam

May, 2004

Name ID

You must show all work to receive full credit.

Do 5 problems out of 6 problems

1. The population dynamics of a species is governed by the discrete model

$$x_{n+1} = x_n[1 + r(1 - x_n)]$$

where $r > 0$.

(a) Find the nonnegative fixed points of period one and determine their stability.

(b) Find the value for r at which a period-doubling bifurcation occurs.

(**Hint:** you may want to use the fact $f(f(x)) - x = -rx(x-1)[r^2x^2 - (r^2 + 2r)x + r + 2]$ where $f(x) = x[1 + r(1 - x)]$.)

2. A certain species of fish can be divided into three age groups, each one year long. The Leslie matrix for the female portion of the population is given by

$$L = \begin{pmatrix} 0 & 3 & 5 \\ 0.3 & 0 & 0 \\ 0 & 0.8 & 0 \end{pmatrix}$$

(a) Find the long-term growth rate and the long-term distribution of the age classes of the population.

Hint: Use the following results produced by Maple:

A:=matrix([[0,3,5],[0.3,0,0],[0,0.8,0]]): eigenvals(A);

1.3400, -0.6700+0.6683I, -0.6700-0.6683I

eigenvects(A);

$[1.3400, 1, \{[3.5808, 0.8017, 0.4786]\}]$

$[-0.6700 + 0.6683I, 1, \{[4.3678 - 0.7635I, -1.1513 - 0.8065I, 0.2075 + 1.1700I]\}]$

$[-0.6700 - 0.6683I, 1, \{[4.3678 + 0.7635I, -1.1513 + 0.8065I, 0.2075 - 1.1700I]\}]$

(b) Determine a sustainable harvesting policy: harvest the oldest age class only.

(**Hint:** use the formula $(1-d_1)[b_1 + b_2c_1(1-d_2) + b_3c_1c_2(1-d_2)(1-d_3)] = 1$.)

(c) Describe how you determine the long-term distribution of the age classes of the population if the sustainable harvesting policy in (b) is applied.

3. Consider the Hénon map

$$x_{n+1} = \alpha + 0.3y_n - x_n^2, \ y_{n+1} = x_n.$$

This system has a chaotic attractor for $\alpha = \alpha_0 = 1.4$. Use the OGY method to control the chaos to a fixed point of period one:

(a) Find the matrix that is used to determne the chaos control;

(b) Describe how you use the matrix found in (a) to determine the parameters in the chaos control.

4. Consider the Lotka-Volterra competition model

$$\dot{x} = x(2 - \alpha x - y), \ \dot{y} = y(1 - x - y)$$

where the parameter α satisfies the condition $0 < \alpha < 2$.

(a) Find all critical points and their local stability.

(b) Show that the point $(\frac{2}{\alpha}, 0)$ is a global attractor for all positive solutions.

5. Consider the predator-prey system

$$\dot{x} = x(\beta - x) - \frac{xy}{1+x}, \ \dot{y} = y(\frac{x}{1+x} - \frac{1}{2})$$

where the parameter β satisfies the condition $\beta > 3$.

(a) Find all critical points and their local stability.

(b) Show that this system has a limit cycle in the first quadrant.

6. Consider the Duffing system given by

$$\dot{x} = y, \quad \dot{y} = x - x^3$$

(a) Find the Hamiltonian of the system.

(b) Find the equation that describe the homoclinic cycles in the system.

(c) Describe solution orbits inside the homoclinic cycles.