Algebra Qualifying Exam, Oct. 2005

1. **Group theory** Do three problems out of the six group theory problems.

   (a) Let G be a group.

   i. Let H be a subgroup of G. Prove that H is normal in G if and only if H is a union of conjugacy classes of G.

   ii. Suppose that a group G has n conjugacy classes. Use part (a) to find an upper bound for the number of normal subgroups of G.

   (b) Let $p < q$ be distinct prime numbers such that $p \not| q - 1$. Prove that G is cyclic.

   (c) Let G be a group and let $[G, G]$ be the subgroup of G **generated** by the set $\{aba^{-1}b^{-1} : a, b \in G\}$.

   i. Prove that $G/[G, G]$ is Abelian.

   ii. Let $f : G \to K$ be a group homomorphism. Prove that $f(G)$ is Abelian if and only if $[G, G] \subseteq Ker(f)$.

   (d) Let $(G, \times)$ be a group with identity e under the operation $\times$. Let $a \in G$. Define the binary operation $\cdot$ on G as follows: for all $x, y \in G$, let $x \cdot y = x \times a \times y$.

   i. Show that $\langle G; \cdot \rangle$ is a group.

   ii. Show that $\langle G; \cdot \rangle$ and $\langle G; \times \rangle$ are isomorphic by providing a specific isomorphism $f : G \to G$.

   (e) Consider the of two-by-two invertible matrices over the real numbers $\mathbf{R}$ consisting of $G = \{ \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} : c \in \mathbf{R}^+,\ d \in \mathbf{R} \}$

   i. Show that G is a group.

   ii. Find a normal subgroup N of G and a subgroup H of G such that $N \cap H$ is the identity subgroup and $G = NH$.

   iii. Provide a brief justification for your answer to the following question: Is G isomorphic to $N \times H$?

   (f) Let G be a finite group.

   i. State the Class Equation for G, an equation involving $|G|$ and cardinalities of conjugacy classes of G.

   ii. Let p be a prime number and n be a positive integer. Use the Class Equation to prove that if $|G| = p^n$, then G has a non-trivial center.

2. **Ring theory** Do two problems out of the four problems below. Let K be a ring below. Recall that an ideal I of K is said to be *prime* if for all $s, t \in A$ we have $st \in I$ implies at least one of s, t is in I. An ideal J of K is said to be *maximal* if J is properly contained in K and J is not properly contained in any proper ideal of K.

   (a) Prove that if K is a commutative ring with identity and I is an ideal of K, then K/I is a field if and only if I is a maximal ideal of K.

   (b) Let K, L be rings and let $f : K \to L$ be a surjective ring homomorphism. Construct a bijection between the set of all ideals of L and $\{I \leq K : Ker(f) \subseteq I\}$, the set of ideals of K which contain $Ker(f)$.

   (c) If you choose this problem, do both parts.

      i. Prove that in a commutative ring with identity a maximal ideal is a prime ideal.

      ii. True or false? A prime ideal is a maximal ideal. If true, provide a proof; if false a specific counterexample.

   (d) Prove that $Z[\sqrt{-5}]$ is not a Unique Factorization Domain.

3. **Field theory** Do two out of four of the problems below.

   (a) Consider the polynomial $q(x) = x^4 - 2 \in Q[x]$. Let S be the splitting field of $q(x)$ over Q and let $G = Gal(S/Q)$, the Galois group of $p(x)$ over Q.

      i. Determine the splitting field S of $q(x)$.

      ii. Determine $[S : Q]$, the vector space dimension of S over Q.

      iii. Show that G is not Abelian.

   (b) Let F be a finite field.

      i. Prove that the characteristic of F is a prime number.

      ii. Prove that $|F| = p^n$, where p is the characteristic of F.

   (c) Suppose that $E \leq F \leq K$ is a chain of field extensions and that $[F : E] = m$, that $[E : F] = n$ and that $\{v_1, \ldots, v_m\}$ is a basis for E over F and $\{w_1, \ldots w_n\}$ is a basis for K over F. Prove that $\{v_i w_j : i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}\}$ is a basis for K over E.

   (d) Suppose that E, F are fields and that $[E : F] = n$, where n is finite.

      i. Prove that E is algebraic over F.

      ii. Suppose that $[E : F] = p$ is prime. Let α be an element of E. Prove that the degree of $m_\alpha(x) \in F[x]$, the minimal polynomial of α, is either 1 or p.