10/26/05 **Fall Analysis Qualifier**

Name:

The test consists of three parts.

1. The first part consists of 5 True/False questions. This part is **mandatory**.
2. You will be awarded credit for 3 out of 5 questions in the second part, and 3 out of 5 questions in the third part. In the following boxes, *please circle the 3* that you would like us to grade.

Please do **all** of the following:

Part I	(a)	(b)	(c)	(d)	(e)	*Total*
Max. pts	5	5	5	5	5	25
Grade						

Please **circle exactly 3** of the following:

Part II	1	2	3	4	5	*Total*
Max. pts	25	25	25	25	25	75
Grade						

Please **circle exactly 3** of the following:

Part II	1	2	3	4	5	*Total*
Max. pts	25	25	25	25	25	75
Grade						

Note: Please note that a complete and correct solution will carry far more weight than several sparsely supported "solution sketches".

FINAL SCORE (out of 175): **PASS** **FAIL**

PART I. State whether each of the following statements is True (T) or False (F). Support your assertion with a proper justification. You will receive 2 points for the correct choice and 3 points for the justification.

(a) Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[0, 1]$. If $f_n \to f$ uniformly on $[0, 1]$, then f is also Riemann integrable.
T F

(b) There is a monotone increasing function on $[0, 1]$ that is not absolutely continuous. T F

(c) Every non-empty perfect nowhere dense subset of the reals must have Lebesgue measure zero. T F

(d) Let f_n, f be finite, real valued measurable functions on $[0, 1]$ equipped with Lebesgue measure. Then $f_n \to f$ in measure implies $f_n \to f$ a.e T F

(e) There is a function of bounded variation on $[0, 1]$ which is non-differentiable precisely on the Cantor ternary set. T F

PART II.

1. Clearly state the Axiom of Choice. Show that the Axiom of Choice follows from the Well Ordering Principle (every set can be well-ordered).

2. Clearly state the Baire Category Theorem. Show that a dense $\mathcal{G}_\delta$ set in a complete metric space is residual, i.e. it must be the complement of a first-category set.

3. Clearly define an absolutely continuous function on $[a, b]$. Show that if f is absolutely continuous on $[a, b]$ then for every $E \subseteq [a, b]$, of Lebesgue measure zero (i.e. $\lambda(E) = 0$), we must have $\lambda(f(E)) = 0$.

4. Clearly define a separable metric space. Let $\mathcal{C}[a, b]$ denote the space of continuous functions on $[a, b]$. Is $\mathcal{C}[a, b]$ separable? Sketch a proof.

5. Let X, Y be normed linear spaces and let $T : X \to Y$. Clearly define what it means for T to be a bounded linear operator. Prove that a bounded linear operator is continuous.

PART III.

1. Suppose $f : \mathbf{R} \to \mathbf{R}$. Then there is a constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbf{R}$ iff f is absolutely continuous and $|f'| \leq M$ Lebesgue a.e.

2. Let $(X, \mathcal{B}, \mu)$ be a measure space, $f \in \mathcal{L}^1(\mu)$. Define

$$\nu(A) = \int_A f d\mu, \ A \in \mathcal{B}$$

Show that ν is a signed measure on X and find a Hahn decomposition for ν.

3. Let (X, ρ) be a complete metric space and suppose $f : X \to X$ has the property that

$$\rho(f(x), f(y)) < C\rho(x, y)$$

for some $0 < C < 1$ and for all $x, y \in X, \ x \neq y$. Show that f has a unique fixed point.

4. A sequence (f_n) in $\mathcal{L}^p([0,1])$ is said to converge weakly to a function $f \in \mathcal{L}^p([0,1])$ if $\int f_n g \to \int fg$ for all $g \in \mathcal{L}^q([0,1])$, where p, q are conjugates. Show that every orthonormal sequence converges weakly to 0 in $\mathcal{L}^2([0,1])$.

5. Let $\mathcal{C}[0,1]$ denote the space of continuous functions on $[0,1]$. For a fixed choice of $t_1, t_2, \ldots t_n \in [0,1]$ and $\lambda_1, \ldots \lambda_n \in \mathbf{R}$, we may define $F : \mathcal{C}[0,1] \to \mathbf{R}$ as

$$F(f) = \sum_{i=1}^{n} \lambda_i f(t_i).$$

Show that F is a linear functional on $\mathcal{C}[0,1]$ and find the norm of F.