Ph.D. Qualifying Examination in Algebra

Department of Mathematics
University of Louisville
Spring 2005

1 Groups

Do any two problems from this section.

1. Prove that a group that has only a finite number of subgroups must be finite.

2. Prove that $G = \{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : \ a, b \text{ real and } a > 0\}$ is a group under matrix multiplication. Show that G contains a subgroup H such that H is isomorphic to the group $\mathbb{R}^+$ of positive real numbers under multiplication.

3. Classify up to isomorphism all groups with 225 elements.

4. Do all four parts to receive full credit. Let G be a finite group.
   (a) What is a *subnormal series* for G?
   (b) What is a *composition series* for G?
   (c) What condition or conditions must G satisfy to be a *solvable group*?
   (d) Show that S_3 is solvable.

2 Rings

Do any two problems from this section.

1. Let $x^4 - 16$ be an element of the polynomial ring $E = \mathbb{Z}[x]$ and use the bar notation to denote passage to the quotient ring $\mathbb{Z}[x]/(x^4 - 16)$.

(a) Find a polynomial of degree ≤ 3 that is congruent to $7x^{13} - 11x^9 + 5x^5 - 2x^3 + 3$ modulo $(x^4 - 16)$.

(b) Prove that $\overline{x-2}$ and $\overline{x+2}$ are zero divisors in $\overline{E}$.

2. Prove that every Euclidean domain is a principal ideal domain (PID).

3. An integral domain D is said to be **Artinian** if for any descending chain of ideals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq ...$$

of D, there is an integer n such that $I_i = I_n$ for all $i \geq n$. Prove that an integral domain D is Artinian if and only if it is a field. (Artinian rings are also referred to as rings satisfying the *Descending Chain Condition.*)

4. Prove the Third Isomorphism Theorem for Rings. That is prove that if R is a ring and I and J are ideals of R with $I \leq J$, then $R/J \approx (R/I)/(J/I)$.

3 Fields

Do two problems from this section with one of the problems being either 1 or 2.

1. Let $p(x) \in \mathbb{Q}[x]$ be a polynomial of degree 3.

   (a) Prove that if $p(x)$ is irreducible, then its Galois group is isomorphic to either A_3 or S_3.

   (b) Provide an example of a degree three polynomial $p(x) \in \mathbb{Q}[x]$ whose Galois group is not isomorphic to A_3 or S_3.

2. Let $q(x) = x^7 - 5 \in Q[x]$ and let S be a splitting field for $q(x)$.

   (a) Determine $[S : \mathbb{Q}]$, the dimension of S over $\mathbb{Q}$.

   (b) Determine two distinct subfields K and L of S such that K and L are properly contained in S and both properly contain $\mathbb{Q}$. Demonstrate that K and L are not equal and that they are indeed proper and properly contain $\mathbb{Q}$.

3. Let F be a finite field of characteristic p. Consider the Frobenius map $g : F \to F$, given by $g(x) = x^p$ for $x \in F$.

   (a) Prove that g is an automorphism of F.

   (b) Determine the set $\{\alpha : g(\alpha) = \alpha\}$, providing complete justification for your solution.

4. Let E be a finite extension of F. Prove that E is algebraic over F.