5/13/05 **Spring Analysis Qualifier**

Name:

The test consists of three parts.

1. The first part consists of 5 True/False questions. This part is mandatory.

2. You will be awarded credit for 3 out of 5 questions in the second part, and 3 out of 5 questions in the third part. In the following boxes, *please circle the 3* that you would like us to grade.

Please do **all** of the following:

Part I	(a)	(b)	(c)	(d)	(e)	*Total*
Max. pts	5	5	5	5	5	25
Grade						

Please **circle exactly 3** of the following:

Part II	1	2	3	4	5	*Total*
Max. pts	25	25	25	25	25	75
Grade						

Please **circle exactly 3** of the following:

Part II	1	2	3	4	5	*Total*
Max. pts	25	25	25	25	25	75
Grade						

Note: Please note that a complete and correct solution will carry far more weight than several sparsely supported "solution sketches".

FINAL SCORE (out of 175): **PASS** **FAIL**

PART I. State whether each of the following statements is True (T) or False (F). Support your assertion with a proper justification. You will receive 2 points for the correct choice and 3 points for the justification.

(a) There exists a monotone function $f : [0,1] \to [0,1]$ such that f is discontinuous precisely at points in the Cantor ternary set.

T F

(b) Let $\{f_n\}$ be a sequence of non-Riemann integrable functions on $[0,1]$. If $f_n \to f$ uniformly on $[0,1]$, then f is also not Riemann integrable. T F

(c) There is a dense open subset A of the reals, such that A has an uncountable complement. T F

(d) There exists a monotone function $f : [0,1] \to [0,1]$ such that f is non-differentiable precisely at points in a Cantor set of positive Lebesgue measure. T F

(e) Every closed and bounded subset of a complete metric space is compact. T F

PART II.

1. Clearly state the Cauchy-Schwarz inequality. Show that if $\sum_{n=1}^{\infty} a_n^2$ converges absolutely, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ must also converge absolutely.

2. Clearly state the Heine-Borel Theorem. For a bounded open set A of real numbers, give an explicit construction of an open cover that has no finite subcover.

3. Clearly state Fatou's lemma and the Dominated Convergence Theorem. Now use Fatou's lemma to prove the Dominated Convergence Theorem.

4. Clearly state the Riesz Representation Theorem for $\mathcal{L}^p[0,1]$ for $1 < p < \infty$. For $g \in \mathcal{L}^1[0,1]$, prove that $T(f) = \int_0^1 fg \, dx$ defines a bounded linear functional on $\mathcal{L}^\infty[0,1]$ and find $||T||$.

5. For a continuous function f on $[0,1]$, clearly define its Riemann integral as a limit of partial sums using uniform partitions. Show that
$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} \right) = \ln 2.$$

PART III.

1. Suppose $f \in \mathcal{L}^1[0,1]$ and set
$$F(x) = \int_0^x f(t)dt.$$
Prove that F is of bounded variation on $[0,1]$.

2. Let $f : \mathbf{R} \to \mathbf{R}$. Show that f is discontinuous on a set of first category in $\mathbf{R}$ if and only if f is continuous at a dense set of points.

3. Let $\mathcal{H}$ be a separable Hilbert space, and let $\{\phi_k\}$ be an orthonormal set in $\mathcal{H}$. Show that for any $x \in \mathcal{H}$, we have
$$\sum_{k=1}^{\infty} | < x, \phi_k > |^2 \leq ||x||^2.$$
State a sufficient condition for equality.

4. Suppose that ν is a σ-finite signed measure and μ is a σ-finite measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$. Show that if $g \in \mathcal{L}^1(\nu)$, then
$$g\frac{d\nu}{d\mu} \in \mathcal{L}^1(\mu) \text{ and } \int g d\nu = \int g \frac{d\nu}{d\mu} d\mu,$$
where $\frac{d\nu}{d\mu}$ is the Radon-Nikodym derivative of ν with respect to μ.

5. Let C denote the Cantor ternary set in $[0,1]$. Show that
$$C - C = \{a - b \mid a, b \in C\} = [-1, 1].$$