

COMBINATORICS QUALIFYING EXAM  
May 2005

This examination consists of two parts, A and B. Part A contains **six** problems of which you must select four to do. Part B contains **three** problems of which you must select two to do. Each problem in part A is worth 15 points and each problem in part B is worth 20 points. Only hand-in your solutions to four problems from part A and two from part B. Begin each problem on a new sheet of paper and be sure to label each page of your work with the problem number and your name.

In each question, if you appeal to a theorem within your solution, you must carefully state that theorem. All graphs, unless otherwise stated, should be understood to be finite and simple.

PART A: (15 points each) *Do any four.*

**Time: 2 hours.**

PROBLEM A1: Let  $T$  be a finite tree in which there are no vertices of degree two. Recall that a vertex of degree one in a tree is called a *leaf*.

- a) Prove that at least half of the vertices of  $T$  are leaves.
- b) Prove that if  $T$  is sufficiently large, then it must contain a set  $S$  of 100 leaves with the property that all distances (in  $T$ ) between elements of  $S$  are equal modulo 3.

PROBLEM A2: Prove that there are no 4-regular bipartite planar graphs.

PROBLEM A3: A set of integers  $A$  is *fat* if each of its elements is  $\geq |A|$ . For example, the empty set and  $\{5, 7, 91\}$  are fat, but  $\{3, 5, 10, 14\}$  is not. Let  $f_n$  denote the number of fat subsets of  $\{1, \dots, n\}$ .

- a) Find a recurrence relation for  $f_n$ .
- b) Find an explicit formula for  $f_n$ . Justify your answer.

PROBLEM A4: Recall that a *total* order is a partial order in which all pairs are comparable. Suppose that  $P_1$  and  $P_2$  are two total orders on a set of  $n^2 + 1$  elements. Show that there is a subset of size  $n + 1$  on which  $P_1$  and  $P_2$  totally agree or totally disagree.

PROBLEM A5: Use exponential generating functions to find the number of ways to distribute  $n$  distinguishable balls to five different boxes with a positive even number of balls distributed to box 5.

PROBLEM A6: Find the minimum number of edges whose removal from  $K_6$  leaves a planar graph. Justify your claim.

PART B: (20 points each) *Do any two.*

**Time: 1 hour and 20 minutes.**

PROBLEM B1:

- a) Find the number of ways of giving  $3n$  different toys to Maddy, Jimmy, and Tommy so that Maddy and Jimmy together get  $2n$  toys.
- b) Find the number of solutions in nonnegative integers of the equation

$$a + b + c + d + e + f = 20$$

in which no variable is greater than 8.

PROBLEM B2: Prove the given statement or provide a counterexample. Justify your claims.

- a) Every connected graph with at least three vertices has at least two vertices whose removal leaves a connected graph.
- b) If a graph is cubic and has a hamiltonian path, then its edge-chromatic number is three.
- c) If  $G(V, E)$  is a finite simple graph, then  $\{(e, f) \in E \times E: e \text{ and } f \text{ are equal or lie on a common cycle}\}$  is an equivalence relation on the edges of  $G$ .

PROBLEM B3: Consider a finite collection of lines drawn in the plane so that no two lines are parallel and no three lines share a point. Consider their points of intersection as the vertices of a graph and the segments between neighboring intersection points as edges of our graph. Prove that this resulting planar graph is 3-colorable.