

10/24/2006 Analysis Qualifier

The test consist of 3 parts.

Do all five of T/F. They are worth 10 pts each. 3 points for getting T/F correct and 7 points for a correct explanation.

Do 3 of five from section 2. Clearly, state which problems you want graded. Each problem is worth 25 points.

Do 3 of five from section 3. Clearly state which problems you want graded. Each problem is worth 25 points.

Note: A complete and correct solution will carry far more weight than several sparsely supported "solution sketches".

Part I. Decide if the following statements are True or False. You will receive 2 points for deciding if the statement is T/F and 3 points for a correct explanation.

1. Suppose $f_n : [0, 1] \rightarrow \mathbb{R}$ and $\{f_n\}$ converges uniformly to the zero function. Then, some f_n must be Riemann integrable.
2. There are disjoint sets $A, B \subseteq \mathbb{R}$ both of which are dense in \mathbb{R} and both of them have positive measure.
3. Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is such that $|f(x) - f(y)| \leq |x - y|$ for all $x, y \in [0, 1]$. Then, f is differentiable a.e.
4. Every infinite bounded subset of $C([0, 1])$ (the space of continuous functions on $[0, 1]$ endowed with the sup norm) has a limit-point.
5. If $f : [0, 1] \rightarrow \mathbb{R}$ is continuous and of bounded variation and $f'(x) = 0$ for a.e. $x \in [0, 1]$, then f is constant.

Part II. Each of the following problem is worth 25 points. Do 3 of the following. Clearly state which problems you would like graded.

1. State the definition of compactness. Prove that $M \subseteq \mathbb{R}$ is compact iff every infinite subset of M has a limit-point in M .
2. State a version of the Baire category theorem. Use it to show that the set of irrational numbers is not F_σ .
3. State a necessary and sufficient condition for a function $f : [0, 1] \rightarrow \mathbb{R}$ to be Riemann integrable. Give an example of a closed set whose characteristic function is not Riemann integrable. Explain why your example works.
4. Prove that the set of Borel sets is the smallest σ -algebra containing all sets of the form $[a, \infty)$.
5. Suppose $f : [0, \infty) \rightarrow \mathbb{R}$ is twice differentiable and $\lim_{x \rightarrow \infty} f''(x) = 0$. Determine

$$\lim_{x \rightarrow \infty} [f(x - 5) + f(x + 5) - 2f(x)].$$

Give a complete justification for your answer, clearly stating which theorems you are using.

Part III. Each of the following problem is worth 25 points. Do 3 of the following. Clearly state which problems you would like graded.

1. Let $A \subseteq \mathbb{R}$ be such that $\lambda(A) > 0$. (λ denotes the Lebesgue measure.) Show that there is a closed interval I such that $9\lambda(A \cap I) > \lambda(I)$.
2. Suppose $f : [0, 1] \rightarrow \mathbb{R}$ is an L^1 function. For a Borel set $B \subseteq \mathbb{R}$, define

$$\mu(B) = \int_{[0,1]} f d\lambda.$$

(λ denotes the Lebesgue measure.) Explain why μ is absolutely continuous with respect to λ . Give the Hahn-Decomposition of μ .

3. State the Lebesgue dominated convergence theorem. Use it prove that if $\{f_n\}$ is a sequence of Lebesgue integrable functions defined on $[0, 1]$ which converges uniformly to f , then f is Lebesgue integrable and $\lim_{n \rightarrow \infty} \int_0^1 f_n = \int_0^1 f$.
4. For each $n \in \mathbb{N}$, define an operator T_n on $C([0, 1])$, the space of continuous functions endowed with the sup norm, as follows.

$$T_n(f) = \frac{1}{n} \sum_{i=0}^{n-1} f(i).$$

Prove that each T_n is a continuous linear operator on $C([0, 1])$. Find an operator T so that $\{T_n\}$ converges to T .

5. Prove that $L^\infty([0, 1])$ has an uncountable set which has no accumulation point and conclude that it is not separable.