Mathematical Modeling Qualifier Exam

October, 2006

Name ______________________________ **ID** ____________________

There are two parts in this Exam.

Do any 3 problems out of the 4 problems in Part I, and do any 3 problems out of the 4 problems in Part II.

You must show all work to receive full credit. If you do more problems than what are required, you MUST indicate what problems you want to be graded. Otherwise, they may be randomly chosen to grade.

Part I:

1. Consider the equation of a nonlinear pendulum:

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta,$$

where θ is the angle from the vertical downward direction. ($\theta = 0$ is the vertical direction.)

(a) What are the two equilibrium positions?

(b) Derive an expression for conservation of energy.

(c) Derive an integral formula for the period of the nonlinear pendulum.

2. If a spring-mass system has a friction force proportional to the cube of the velocity, then

$$m\frac{d^2x}{dt^2} + \sigma\left(\frac{dx}{dt}\right)^3 + kx = 0,$$

where $x = x(t)$ is the position of the mass at time t, m, σ and k are positive constants.

(a) Let $v = \frac{dx}{dt}$ and sketch the solution in the phase plane.

(b) Let $v = \frac{dx}{dt}$ and solve the problem exactly.

(c) Use the results in (a) and (b) to predict how the system behave.

3. Consider the traffic model

$$\frac{\partial\rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial\rho}{\partial x} = 0,$$

with the initial condition

$$\rho(x,0) = \begin{cases} 4 & x < 0 \\ 0 & 0 < x < 4 \\ 2 & x \geq 4. \end{cases}$$

Suppose that the traffic flux is given by the function $q(\rho) = 12\rho(1 - \frac{\rho}{4})$.

(a) Sketch the characteristics and the shocks.

(b)Find formulas for the density $\rho(x,t)$ and the shock $x_s(t)$.

4. Consider the nonlinear car-following model

$$\frac{d^2 x_n(t+T)}{dt^2} = -\lambda \frac{\frac{dx_n(t)}{dt} - \frac{dx_{n-1}(t)}{dt}}{|x_n(t) - x_{n-1}(t)|}, \quad n = 0, 1, 2, ..., N-1,$$

where $x_n(t)$ is the location of the n^{th} car, N is the number of cars, $T > 0$ is a response time delay, and $\lambda > 0$ is a constant.

(a) Assume that $v_n(t) = \frac{dx_n(t)}{dt} = 0$, $(n = 0, 1, 2, ..., N-1)$, for $t < 0$, and $v_0(t)$ is known for $t > 0$. Formulate a general procedure to solve this nonlinear time-delay equation. **Do not evaluate the integrals.**

(b) Under some assumptions, the car-following model may be simplified to a linear model

$$\frac{d^2 x_n(t+T)}{dt^2} = -\lambda \left(\frac{dx_n(t)}{dt} - \frac{dx_{n-1}(t)}{dt} \right).$$

Consider three cars only. Assume that $T = 1$, $\lambda = 2$. Let $v_n(t) = \frac{dx_n(t)}{dt}$. If $v_n(t) = 0$ for $t < 0$, n=0,1,2, and $v_0(t) = t + t^2/6$ for $t > 0$, compute $v_1(2.5)$ and $v_2(2.5)$.

Part II:

1. Consider the predator-prey system

$$x' = 2x - xy,$$
$$y' = -3y + xy.$$

(a) Find all critical points.

(b) Discuss the linearlized system in a neighborhood of each critical point. (Describe its type and stability.)

(3) Plot a phase portrait for the nonlinear system.

(4) Interpret the solution in terms of species behavior.

2. (a) Prove the system

$$x' = y + x\left(\frac{1}{2} - x^2 - y^2\right),$$
$$y' = -x + y\left(1 - x^2 - y^2\right)$$

has a stable limit cycle.

(b) Find the Hamiltonian of the system

$$\dot{x} = y,$$
$$\dot{y} = x - x^3$$

and sketch a phase portrait.

3. Consider the following one-parameter system of differential equations:

$$\dot{x} = -x^4 + 5\mu x^2 - 4\mu^2,$$
$$\dot{y} = -y.$$

Find the critical points, plot phase portraits, and sketch the corresponding bifurcation diagram.

4. (a) Obtain a Poincaré map for the system

$$\dot{x} = \mu x + y - x\sqrt{x^2 + y^2},$$

$$\dot{y} = x + \mu y - y\sqrt{x^2 + y^2}$$

on the Poincaré section $\Sigma = \{(x, y) \in R^2 : 0 \leq x < \infty, y = 0\}$.

(b) Use the characteristic multiplier to determine the stability of the limit cycle.