Ph.D. Qualifying Examination in Algebra

Department of Mathematics

University of Louisville

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1 Groups

Do any two problems from this section.

1. Let G be a group of order $5^2 7^2$.

   (a) Using the Sylow Theorems, show that G is Abelian.

   (b) Determine all isomorphism types of Abelian groups of order $5^2 7^2$.

2. Let G be a finite group with exactly three conjucacy classes. Show that exactly one of the following possibilities holds:

   (a) $|G| = 3$.

   (b) $|G| = 6$ and G is non-Abelian.

3. Prove that an infinite group is cyclic if and only if it is isomorphic to each of its proper non-trivial subgroups.

4. Let G be a finite group.

   (a) State the Class Equation for G in a form that involves the center $Z(G)$ of G and the index of centralizers of elements of G.

   (b) Determine the conjugacy class of S_4 and their orders. Then verify the Class Equation, given in part (a), is valid for S_4.

   (c) Let p be a prime number and let G be a non-trivial p-group. Use the Class Equation to show that G has a non-trivial center.

5. Prove that if a group G is nilpotent, then it is solvable.

2 Rings

Do any two problems from this section.

1. Do both.

   (a) Prove that if F is a field, then the polynomial ring $F[x]$ is a Euclidean domain. That is, $F[x]$ is an integral domain and there exists a function $\phi : F[x] - \{0\} \to \mathbb{N}$ such that if $f, g \in F[x]$ and $g \neq 0$, then there exist $q, r \in F[x]$ such that $f = qg + r$ with $r = 0$ or $\phi(r) < \phi(g)$.

   (b) Prove or disprove that if R is an integral domain and A is a proper ideal of R, then R/A is an integral domain.

2. Do both.

   (a) Show that $\mathbb{Z}[i]/(3+i)$ is isomorphic to $\mathbb{Z}/10\mathbb{Z}$.

   (b) Is $(3+i)$ a maximal ideal of $\mathbb{Z}[i]$? Give a reason for your answer.

3. Let R, S be rings and let $f : R \to S$ be a surjective ring homomorphism (that is, $f(R) = S$). Let $Ker(f)$ be the kernel of f.

   Define $f^* : R/Ker(f) \to S$ as follows: $f^*(a + Ker(f)) = f(a)$.

   (a) Show that f^* is well-defined.

   (b) Show that $f^* : R/Ker(f) \to S$ is a bijection.

   (c) Show that $f^* : R/Ker(f) \to S$ is a ring homomorphism.

3 Fields

Do two problems from this section with one of the problems being problem 3

1. Do both.

   (a) Prove that $F = \{\begin{bmatrix} a & b \\ 2b & a \end{bmatrix} : a, b \in \mathbb{Q}\}$ is a field under matrix addition and multiplication.

   (b) Prove that F is isomorphic to the field $\mathbb{Q}(\sqrt{2})$.

2. Given $f(x) = x^3 + x^2 + 1 \in \mathbb{Z}_2[x]$.

   (a) Show that $f(x)$ is irreducible over $\mathbb{Z}_2$.

   (b) Let $F = \mathbb{Z}_2[x]/\langle f(x)\rangle$ and α be a zero of $f(x)$ in F. Show that for each nonzero element β of F, there exists a polynomial $q(x) \in Z_2[x]$ of degree at most 2 such that $\beta = q(\alpha)$.

3. Consider $p(x) = x^3 + 5 \in Q[x]$. Let S be the splitting field of $p(x) \in Q[x]$ and assume S is contained in $\mathcal{C}$, the field of complex numbers. You will be graded on the correctness and thoroughness of your explanations: provide relevant theorems, facts etc. to support each of your assertions.

   (a) Determine $[S : Q]$, showing all details.

   (b) Determine $Gal(S/Q)$, up to isomorphism. Provide thorough explanation, citing relevant theorems, facts, etc. to support your conclusions.

   (c) Explain why S contains exactly one subfield J, with $Q \leq J \leq S$, such that $[J : Q] = 2$.