5/22/06 **Spring Analysis Qualifier**

Name:

The test consists of three parts.

1. The first part consists of 5 True/False questions. This part is **mandatory**.
2. You will be awarded credit for 3 out of 5 questions in the second part, and 3 out of 5 questions in the third part. In the following boxes, *please circle the 3* that you would like us to grade.

Please do **all** of the following:

Part I	(a)	(b)	(c)	(d)	(e)	*Total*
Max. pts	5	5	5	5	5	25
Grade						

Please **circle exactly 3** of the following:

Part II	1	2	3	4	5	*Total*
Max. pts	25	25	25	25	25	75
Grade						

Please **circle exactly 3** of the following:

Part II	1	2	3	4	5	*Total*
Max. pts	25	25	25	25	25	75
Grade						

Note: Please note that a complete and correct solution will carry far more weight than several sparsely supported "solution sketches".

FINAL SCORE (out of 175): **PASS** **FAIL**

PART I. Mandatory. State whether each of the following statements is True (T) or False (F). Support your assertion with a proper justification. You will receive 2 points for the correct choice and 3 points for the justification.

(a) Given a fat Cantor set $F \subseteq [0,1]$, there is a function of bounded variation on $[0,1]$ which is non-differentiable precisely on F.
T F

(b) If $C_k \subseteq \mathbf{R}$ is a nested sequence of non-empty, closed sets, then $\bigcap C_k \neq \emptyset$
T F

(c) For any function $f : [0,1] \to [0,1]$, if $f' = 0$ a.e., then f is a constant.
T F

(d) Let f_n, f be finite, real valued measurable functions on $[0,1]$ equipped with Lebesgue measure. Then $f_n \to f$ in $L^1[0,1]$ implies $f_n \to f$ a.e
T F

(e) Suppose $f : \mathbf{R} \to \mathbf{R}$ and $f(x) = x_0$ for all $x \in \mathbf{Q}$. Then f is a constant.
T F

PART II. Please choose 3 out of 5 from the following:

1. Define an F_σ-set. If X is a topological space and $f : X \to \mathbf{R}$, show that the set of points at which f is not continuous is an F_σ-set.

2. For a function $f : [0, 1] \to \mathbf{R}$ define what it means for f to be of bounded variation. Let $f(0) = 0$ and $f(x) = x \sin(1/x),\ x \neq 0$. Show that f is not of bounded variation.

3. Define a separable metric space. Prove that every compact metric space is separable.

4. State the Baire Category Theorem. Use this to prove that $\mathbf{R}$ is uncountable.

5. Define a Hilbert space. For a Hilbert space $\mathcal{H}$ and an orthonormal basis B of $\mathcal{H}$, prove that $||x||^2 = \sum_{x_j \in B} < x, x_j >^2,\ \forall x \in X.$

PART III. Please choose 3 out of 5 from the following:

1. Suppose $A \subseteq \mathbf{R}$ is an uncountable set. Then A has uncountably many accumulation points.

2. Let f be a real-valued function on $[0,1]$, and let λ^* denote the Lebesgue outer measure. Suppose $E \subseteq [0,1]$ and f' exists and is bounded on E by a constant bound M. Prove that $\lambda^*(f(E)) \leq M\lambda^*(E)$.

3. Suppose ν, μ are σ-finite measures on the measurable space $(X, \mathcal{M})$, such that $\nu \ll \mu$. Prove that if $g \in L^1(\nu)$, then $g(d\nu/d\mu) \in L^1(\mu)$ and $\displaystyle\int g d\nu = \int g \frac{d\nu}{d\mu} d\mu$.

4. Let $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Prove that for all $f \in (l^p)^*$ there is a fixed $t \in l^q$ such that $f(s) = \sum s_n t_n$ for all $s \in l^p$.

5. Let $\mathcal{P}$ denote the space of all polynomials on $[0,1]$ with $L^\infty[0,1]$ norm. If $F : \mathcal{P} \to \mathcal{P}$ is defined by

$$F(\sum_{k=0}^{n} a_k x^k) = \sum_{k=0}^{n} a_k x^{k+1},$$

show that F is continuous and find $||F||$.