Preliminary Exam
COMBINATORICS
May, 2006

This examination consists of two parts, A and B. Part A consists of five problems and part B consists of three problems. Each problem in part A is worth 15 points and each problem in part B is worth 20 points. You have to solve any four problems out of part A and any two problems out of part B. Begin each problem on a new sheet of paper, and only write on one side of the paper. Only hand in those selected six problems. You have 3 hours and 30 minutes to complete the exam.

PART A (15 points each) Do any <u>four</u>.

Problem A1.
Let G be a simple graph with n vertices, $n \geq 3$.

(a) Determine, with a proof, all graphs G having the property that $G - e$ is a tree for every edge $e \in E(G)$. Give an example of such a graph of order $n = 5$.

(a) Characterize those graphs G for which $G - e$ is a tree for some edge $e \in E(G)$. Give an example of such a graph of order $n = 5$ different than an example in part (a).

Problem A2.
For a graph G, let $\alpha(G)$ denote the maximum size of an independent set of vertices in G. Suppose that G is a bipartite graph of order $2m$.
Prove: $\alpha(G) = m$ if and only if G has a perfect matching.

Problem A3.
A diameter, *diam*(G), of a graph G is the length of the longest path in G.
$\chi(G)$ is the chromatic number of G.

(a) Prove that $\chi(G) \leq diam(G) + 1$.

(a) Give an example of a graph G for which $\chi(G) = diam(G) + 1$.

(a) Show that the difference between the numbers $diam(G) + 1$ and $\chi(G)$ can be arbitrarily large.

Problem A4.
A caterpillar is a tree having the property that after deleting all leaves (vertices of degree 1) from it, the remaining graph is a path. A diameter of a tree, *diam(T)*, is the length of the longest path.
Show that if T is a caterpillar of order n with *diam(T)* $= k$ ($k < n$), then its independence number $\alpha(G) \geq n - k + 1$.

Problem A5.
Let a_n denote the number of n-digit sequences in which each digit is 0, 1, or –1, with no two consecutive 1s or two consecutive –1s allowed.
Prove that a_n satisfies the recurrence relation
$a_n = 2a_{n-1} + a_{n-2}$, $n \geq 3$, and find a formula for a_n.

PART B (20 points each) Do any <u>two</u>.

Problem B1.

An $n \times n \times n$ cube consists of n^3 unit cubes stacked into a rectangular pile having width, length, and height *n*. Two units cubes are <u>adjacent</u> if they share a 2-dimensional face.

Determine with a proof all values of *n*, $n \geq 2$, for which it is possible to list all unit cubes in such a way that all three conditions are satisfied:

(1) no cube is repeated;
(2) every two consecutive cubes in the listing are adjacent;
(3) the last cube and the first cube in the listing are adjacent.

Problem B2.

(a) Find a formula for the number of solutions of $x_1 + x_2 + \ldots + x_k < n$, where *n*, x_i are positive integers and *k* is fixed.
(b) Find a formula for the number of solutions of $x_1 + x_2 + \ldots + x_k = n$, where $x_i = \pm 1$, *n* and *k* are fixed positive integers.

Problem B3.

Five differently colored dice are thrown simultaneously and the numbers of dots on them are added.

(a) Use the ordinary generating function to find the number of outcomes with the sum of dots equal to 22.
(b) Use the ordinary generating function to find the number of outcomes with the sum of dots equal to 22 and even number of dots on each die.