Name:

Instructions: Clearly mark the answers and show all the work. You may use calculators when appropriate. Solve three problems from each part. Clearly marke the ones you are NOT solving

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PART ONE: Probabillity

Solve three out of the next four problems

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1. (50 points) This question consists of three parts.

   (a) (15 points) Let X_i be a sequence of random variables such that

   $$\lim_{n\to\infty} \frac{VarS_n}{n^2} = 0, \tag{1}$$

   where $S_n = \sum_{i=1}^n X_i$. Show that

   $$\frac{S_n - ES_n}{n} \xrightarrow{P} 0 \quad \text{as } n \to \infty. \tag{2}$$

   (b) (15 points) Now, suppose that we replace the condition (1) with the assumption that the X_i are pairwise uncorrelated and satisfy $\sup_i EX_i^2 < \infty$. Show that the result (2) above also holds under these alternative assumptions.

   (c) (20 points) Assuming that the X_i are independent random variables such that $\sup_i EX_i^4 < \infty$, show that the convergence in probability in (2) can be strengthened to convergence almost surely.

2. (50 points points) Given a probability space $(\Omega, \mathcal{F}_0, P)$, an $\mathcal{F}_0$-measureable random variable X and another σ-field $\mathcal{F} \subset \mathcal{F}_0$, the ***conditional expectation of*** X ***given*** $\mathcal{F}$ is defined to be any random variable Y which is $\mathcal{F}$-measureable and satisfies

$$\int_A X dP = \int_A Y dP$$

for all $A \in \mathcal{F}$.

(a) (15 points) To warm up, consider how this definition relates to the one taught in undergraduate probability. Specifically, suppose that $\Omega_1, \Omega_2, \ldots$ is a finite or infinite partition of Ω into disjoint sets each of which has positive probability (with respect to P), and let $\mathcal{F} = \sigma(\Omega_1, \Omega_2, \ldots)$ the σ-field generated by these sets. Then show that on each Ω_i,

$$E(X \,|\mathcal{F}) = \frac{E(X; \Omega_i)}{P(\Omega_i)}.$$

(b) (15 points) Let $\mathcal{F}_1 \subset \mathcal{F}_2$ be two σ-fields on Ω. Then show that

1. $E(E(X \,|\mathcal{F}_1) \,|\mathcal{F}_2) = E(X \,|\mathcal{F}_1)$, and
2. $E(E(X \,|\mathcal{F}_2) \,|\mathcal{F}_1) = E(X \,|\mathcal{F}_1)$.

(c) (20 points) Let $\Omega = \{a, b, c\}$. Give an example of $P, \mathcal{F}_1, \mathcal{F}_2$, and X in which

$$E(E(X \,|\mathcal{F}_1) \,|\mathcal{F}_2) \neq E(E(X \,|\mathcal{F}_2) \,|\mathcal{F}_1).$$

3. (50 points) This problem consists of two parts.

(a) (25 points) Let X be $N(0,1)$ random variable. Let

$$M(s) = E\{e^{s\,X}\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(s\,x - \frac{x^2}{2}\right) dx.$$

Show that $M(s) = e^{s^2/2}$.

(b) (25 points) Show that for any positive integer n we have $E\{X^{2n+1}\} = 0$ and

$$E\{X^{2n}\} = \frac{(2n)!}{2^n\,n!} = (2n-1)(2n-3)\ldots 3 \cdot 1.$$

(Hint: Note that $e^{s^2/2} = \sum_{k=0}^{\infty} \frac{s^{2k}}{2^k\,k!}$

4. (50 points) This question consists of two parts.

   (a) (25 points) Show that for any c.d.f. F and any $a \geq 0$

$$\int [F(x+a) - F(x)]dx = a$$

   (b) (25 points) Let X be a random variable with range $\{0,1,2,\ldots\}$. Show that if $EX < \infty$ then

$$EX = \sum_{i=1}^{\infty} P(X \geq i)$$

PART TWO: Mathematical Statistics
Solve three out of the next four problems

5. (50 points) Let $U_1, \ldots, U_n$ be i.i.d. random variables having uniform distribution on [0,1] and $Y_n = (\prod_{i=1}^n U_i)^{-1/n}$. Show that

$$\sqrt{n}(Y_n - e) \xrightarrow{D} N(0, e^2)$$

where $e = \exp(1)$.

6. (50 points) Let ϕ be a UMP test of level $\alpha \in (0,1)$ for testing simple hypothesis P_0 vs P_1. If β is the power of the test, show that $\beta \geq \alpha$ with equality if and only if $P_0 = P_1$.

7. (50 points) Let $X_1, \ldots, X_n$ be independently and identically distributed with density

$$f(x, \theta) = \frac{1}{\sigma} \exp\left\{-\frac{x-\mu}{\sigma}\right\}, \ x \geq \mu,$$

where $\theta = (\mu, \sigma^2), -\infty < \mu < \infty, \sigma^2 > 0$.

(a) Find maximum likelihood estimates of μ and σ^2.

(b) Find the maximum likelihood estimate of $P_\theta(X_1 \geq t)$ for $t > \mu$.

8. (50 points) Let $X_1, \ldots, X_n$ be iid from Bernoulli distribution with unknown probability of success $P(X_1 = 1) = p \in (0, 1)$.

   (a) (20 points) Show that $S = \sum_{i=1}^{n} X_i$ is a complete and sufficient statistic.

   $$\text{(Hint: } \sum_{k=0}^{n} \binom{n}{k} g(k) p^k (1-p)^{n-k} = (1-p)^n \left(\sum_{k=0}^{n} \binom{n}{k} g(k) \zeta^k \right) \text{ where } \zeta = \frac{p}{1-p}.\text{)}$$

   (b) (30 points) Find UMVUE for p^m when $m \leq n$ is a positive integer.

Done !