

NAME:

INSTRUCTIONS: Clearly mark the answers and show all the work. You may use calculators when appropriate. Solve three problems from each part. Clearly mark the ones you are NOT solving

PART ONE: PROBABILITY
SOLVE THREE OUT OF THE NEXT FOUR PROBLEMS

1. (50 points) This question consists of three parts.
- (a) (15 points) Let X_i be a sequence of random variables such that

$$\lim_{n \rightarrow \infty} \frac{\text{Var} S_n}{n^2} = 0, \quad (1)$$

where $S_n = \sum_{i=1}^n X_i$. Show that

$$\frac{S_n - ES_n}{n} \xrightarrow{P} 0 \quad \text{as } n \rightarrow \infty. \quad (2)$$

- (b) (15 points) Now, suppose that we replace the condition (1) with the assumption that the X_i are pairwise uncorrelated and satisfy $\sup_i EX_i^2 < \infty$. Show that the result (2) above also holds under these alternative assumptions.

- (c) (20 points) Assuming that the X_i are independent random variables such that $\sup_i EX_i^4 < \infty$, show that the convergence in probability in (2) can be strengthened to convergence almost surely.



2. (50 points) Given a probability space $(\Omega, \mathcal{F}_0, P)$, an \mathcal{F}_0 -measurable random variable X and another σ -field $\mathcal{F} \subset \mathcal{F}_0$, the **conditional expectation of X given \mathcal{F}** is defined to be any random variable Y which is \mathcal{F} -measurable and satisfies

$$\int_A X dP = \int_A Y dP$$

for all $A \in \mathcal{F}$.

- (a) (15 points) To warm up, consider how this definition relates to the one taught in undergraduate probability. Specifically, suppose that $\Omega_1, \Omega_2, \dots$ is a finite or infinite partition of Ω into disjoint sets each of which has positive probability (with respect to P), and let $\mathcal{F} = \sigma(\Omega_1, \Omega_2, \dots)$ the σ -field generated by these sets. Then show that on each Ω_i ,

$$E(X | \mathcal{F}) = \frac{E(X; \Omega_i)}{P(\Omega_i)}.$$

- (b) (15 points) Let $\mathcal{F}_1 \subset \mathcal{F}_2$ be two σ -fields on Ω . Then show that

1. $E(E(X | \mathcal{F}_1) | \mathcal{F}_2) = E(X | \mathcal{F}_1)$, and
2. $E(E(X | \mathcal{F}_2) | \mathcal{F}_1) = E(X | \mathcal{F}_1)$.

- (c) (20 points) Let $\Omega = \{a, b, c\}$. Give an example of $P, \mathcal{F}_1, \mathcal{F}_2$, and X in which

$$E(E(X | \mathcal{F}_1) | \mathcal{F}_2) \neq E(E(X | \mathcal{F}_2) | \mathcal{F}_1).$$



3. (50 points) This problem consists of two parts.

(a) (25 points) Let X be $N(0, 1)$ random variable. Let

$$M(s) = E\{e^{sX}\} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(sx - \frac{x^2}{2}\right) dx.$$

Show that $M(s) = e^{s^2/2}$.

(b) (25 points) Show that for any positive integer n we have $E\{X^{2n+1}\} = 0$ and

$$E\{X^{2n}\} = \frac{(2n)!}{2^n n!} = (2n-1)(2n-3)\dots 3 \cdot 1.$$

(Hint: Note that $e^{s^2/2} = \sum_{k=0}^{\infty} \frac{s^{2k}}{2^k k!}$



4. (50 points) This question consists of two parts.

(a) (25 points) Show that for any c.d.f. F and any $a \geq 0$

$$\int [F(x+a) - F(x)]dx = a$$

(b) (25 points) Let X be a random variable with range $\{0,1,2,\dots\}$. Show that if $EX < \infty$ then

$$EX = \sum_{i=1}^{\infty} P(X \geq i)$$



PART TWO: MATHEMATICAL STATISTICS
SOLVE THREE OUT OF THE NEXT FOUR PROBLEMS

5. (50 points) Let U_1, \dots, U_n be i.i.d. random variables having uniform distribution on $[0,1]$ and $Y_n = (\prod_{i=1}^n U_i)^{-1/n}$. Show that

$$\sqrt{n}(Y_n - e) \xrightarrow{D} N(0, e^2)$$

where $e = \exp(1)$.



6. (50 points) Let ϕ be a UMP test of level $\alpha \in (0, 1)$ for testing simple hypothesis P_0 vs P_1 . If β is the power of the test, show that $\beta \geq \alpha$ with equality if and only if $P_0 = P_1$.



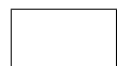
7. (50 points) Let X_1, \dots, X_n be independently and identically distributed with density

$$f(x, \theta) = \frac{1}{\sigma} \exp \left\{ -\frac{x - \mu}{\sigma} \right\}, \quad x \geq \mu,$$

where $\theta = (\mu, \sigma^2)$, $-\infty < \mu < \infty$, $\sigma^2 > 0$.

(a) Find maximum likelihood estimates of μ and σ^2 .

(b) Find the maximum likelihood estimate of $P_\theta(X_1 \geq t)$ for $t > \mu$.



8. (50 points) Let X_1, \dots, X_n be iid from Bernoulli distribution with unknown probability of success $P(X_1 = 1) = p \in (0, 1)$.

(a) (20 points) Show that $S = \sum_{i=1}^n X_i$ is a complete and sufficient statistic.

$$\text{(Hint: } \sum_{k=0}^n \binom{n}{k} g(k) p^k (1-p)^{n-k} = (1-p)^n \left(\sum_{k=0}^n \binom{n}{k} g(k) \zeta^k \right) \text{ where } \zeta = \frac{p}{1-p}.)$$

(b) (30 points) Find UMVUE for p^m when $m \leq n$ is a positive integer.

