Mathematical Modeling Qualifier Exam

May, 2006

Name ____ **ID** ___________________________

There are two parts in this Exam.

Do any 3 problems out of the 4 problems in Part I, and do any 3 problems out of the 4 problems in Part II.

You must show all work to receive full credit. If you do more problems than what are required, you MUST indicate what problems you want to be graded. Otherwise, they may be randomly chosen to grade.

Part I:

1. If a spring-mass system has a friction force proportional to the cube of the velocity, then

$$m\frac{d^2x}{dt^2} + \sigma\left(\frac{dx}{dt}\right)^3 + kx = 0,$$

where $x = x(t)$ is the position of the mass at time t, m, σ and k are positive constants.

(a) Let $v = \frac{dx}{dt}$. Derive a first-order differential equation describing the phase plane. (That is, find a differential equation for v as a function of x.)

(b) Sketch the solution in the phase plane based on the first-order differential equation. Show the critical point and typical trajectories.

2. Consider the differential equation of a nonlinear pendulum

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta,$$

where θ is the angle from the vertical direction. ($\theta = 0$ is the vertical direction.)

(a) Let $\theta(0) = 0$, and $E = \frac{L}{2}\left[\frac{d\theta}{dt}(0)\right]^2$. ($E$ is the total energy when the potential energy is assumed to be 0 at $\theta = 0$.) Derive the energy integral formula for the system.

(b) Estimate the time it takes a pendulum to go completely around, if the energy E is very large ($E >> 2g$).

3. Consider the traffic model

$$\frac{\partial\rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial\rho}{\partial x} = 0,$$

with the initial condition

$$\rho(x,0) = \begin{cases} 3 & x < 0 \\ 0 & 0 < x < 4 \\ 1 & x \geq 4. \end{cases}$$

Suppose that the traffic flux is given by the function $q(\rho) = 2\rho(3-\rho)$.

(a) Sketch the characteristics and the shocks.

(b)Find formulas for the density $\rho(x,t)$ and the shock $x_s(t)$.

4. The traffic flow in a highway with entrances and exits may be modeled as

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = \beta_0.$$

Suppose that $\beta_0 = 1$, $q(\rho) = 4\rho(2-\rho)$ and the initial density is

$$\rho(x,0) = \begin{cases} 2 & x \le 0 \\ 0 & x > 0 \end{cases}$$

(a) Sketch the graphs of characteristics and typical densities for different time t.

(b) Find formulas for the density $\rho(x,t)$ and the characteristics.

Part II:

1. Consider the predator-prey system

$$x' = 4x - x^2 - xy,$$

$$y' = -y - y^2 + xy.$$

(a) Find all critical points and determine their local stability.

(b) Show that the coexistence critical point attracts all positive solutions.
(**Hint:** Use the Lyapunov function $V(x,y) = x - x^* \ln x + y - y^* \ln y - (x^* - x^* \ln x^* + y^* - y^* \ln y^*)$, where (x^*, y^*) denotes the coexistence critical point)

(c) Interpret positive solutions in terms of species behavior.

2. Consider the system

$$x' = x - y - x^3,$$

$$y' = x + y - y^3.$$

(a) This system has only one critical point $(x,y) = (0,0)$. Determine its local stability.

(b) Show that this system has one limit cycle.

(c) Show that this system has at most one limit cycle.

3. Consider the Hamiltonian system

$$\dot{x} = y,$$

$$\dot{y} = x - x^3$$

(a) Find all critical points and determine their local stability

(b) Find the Hamiltonian of the system and sketch a phase portrait.

4. Consider the Hénon map given by

$$x_{n+1} = 1 - \alpha x_n^2 + y_n,$$

$$y_{n+1} = \beta x_n,$$

where $\alpha > 0$ and $|\beta| < 1$.

(a) Find period-one critical points of the system, and describe how you determine their local stability.

(b) Show that when $\alpha = \frac{3(\beta-1)^2}{4}$ for fixed β, two period-two critical points bifurcate from a period-one critical point.