ANALYSIS QUALIFIER

OCTOBER 10, 2007

1. True/False

Determine whether the following statements are true or false and explain how you reached your conclusion. Each problem is worth 10 points. You receive 5 points for a correct answer and 5 additional points for a correct reason.

Problem 1. Every finite subset of a metric space is nowhere dense.

Problem 2. There is a set $S \subset \mathbb{Q}$ such that S is not a Borel set.

Problem 3. If $f : [0, 1] \to \mathbb{R}$ is injective, then there must be a subinterval of $[0, 1]$, on which f is monotone.

Problem 4. Given a differentiable function $f : [0, \infty) \to \mathbb{R}$, assume that

$$\lim_{x\to\infty} f(x) = 0,$$

then

$$\lim_{x\to\infty} f'(x) = 0.$$

Problem 5. $L^2(\mathbb{R}) \subset L^1(\mathbb{R})$

2. Proofs

Each of the following problems is worth 25 points. At most three of your solutions will be graded. If you submit more than three solutions, please indicate which three should be graded.

Problem 1. If $f : [a, b] \to \mathbb{R}$ is a function of bounded variation and there is an $\alpha > 0$ such that $|f(x)| \geq \alpha$ on $[a, b]$, then $1/f$ is of bounded variation on $[a, b]$.

Problem 2. Let λ be Lebesgue measure on $\mathbb{R}$. Show there does not exist a measurable set $A \subset \mathbb{R}$ such that for every interval (a, b), $\lambda(A \cap (a, b)) = (b - a)/2$.

Problem 3. Prove $L^2([0, 1]) \subset L^1([0, 1])$.

Problem 4. Give an example of a set $S \subset \mathbb{R}$ which is both nowhere dense and of positive measure. (Include details of the construction.)

Problem 5. State the Monotone Convergence Theorem and use it to prove Fatou's Lemma.

3. More Proofs

Each of the following problems is worth 25 points. At most three of your solutions will be graded. If you submit more than three solutions, please indicate which three should be graded.

Problem 1. If (X, Σ, μ) is a measure space and $f : X \to (0, \infty)$ is a measurable function, then $1/f$ is measurable.

Problem 2. Let λ be Lebesgue measure on $\mathbb{R}$ and $\varepsilon > 0$. If $S \subset \mathbb{R}$ is bounded and measurable, then $S = K \cup T$ such that K is compact and $\lambda(T) < \varepsilon$.

Problem 3. If $f \in L^1([0, 1])$, then

$$\lim_{n\to\infty} \int_0^1 f(x) \sin nx \, dx = 0.$$

Problem 4. If λ is Lebesgue measure on $\mathbb{R}$ and a measure μ is defined by $\mu(A) = \lambda(A \cap [-1, 1])$, then prove that $\mu << \lambda$ and explicitly find the Radon-Nikodym derivative $d\mu/d\lambda$.

Problem 5. Let $1 \leq p < \infty$. Every Cauchy sequence in $L^\infty([0, 1])$ is a Cauchy sequence in $L^p([0, 1])$.