

COMBINATORICS QUALIFYING EXAM  
October 17, 2007

This examination consists of two parts, A and B. Part A contains **six** problems of which you must select four to do. Part B contains **three** problems of which you must select two to do. Each problem in part A is worth 15 points and each problem in part B is worth 20 points. Only hand-in your solutions to four problems from part A and two from part B. Please do not turn-in more solutions since only the first four solutions from part A will be graded and only the first two solutions from part B will be graded.

Begin each problem on a new sheet of paper and be sure to label each page of your work with the problem number and your name.

In each question, if you appeal to a theorem within your solution, you must carefully state the entire theorem. All graphs, unless otherwise stated, should be understood to be finite and simple.

PART A: (15 points each) Do any four.

Time: 2 hours.

PROBLEM A1: Suppose that the integers 1 through 9 (inclusive) are randomly positioned on a circle.

- Show that the sum of some set of three consecutive integers must be at least 15.
- What can you say about all sums of three consecutive integers, if no such sum exceeds 15?
- Based on your answer to part b), do you believe that some arrangement of these integers exists in which no sum of three consecutive integers exceeds 15? Justify your answer.

PROBLEM A2: Find the number of ways of arranging the 26 letters of the alphabet so that no one of the sequences  $ABC$ ,  $PQRS$ , and  $XYZ$  appears.

PROBLEM A3: Find a generating function whose  $n$ th coefficient gives the number of nonnegative integral solutions of the equation

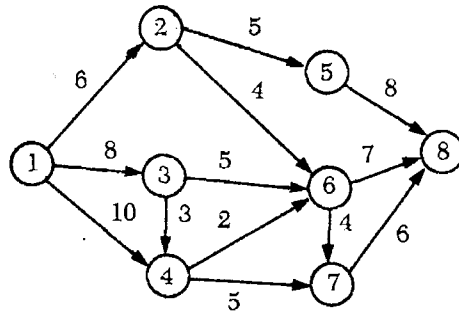
$$z_1 + 4z_2 + 5z_3 + 3z_4 = n.$$

PROBLEM A4: Prove or disprove: The edge-chromatic number of a bipartite graph is equal to its maximum degree.

PROBLEM A5: Suppose that the graph  $G$  is regular of degree  $k \geq 1$  and has at least  $2k + 2$  vertices. Prove that the complement of  $G$  has a hamiltonian cycle.

PROBLEM A6: Consider network shown below with source vertex 1 and sink vertex 8.

- Find a maximum flow.
- Find a minimum cut.
- Explain why the flow and the cut found in parts (a) and (b) are optimal.



PART B: (20 points each) Do any two.

Time: 1 hour and 20 minutes.

PROBLEM B1: Evaluate the given sum. Justify your answer:

a)

$$\sum_{j=0}^r \binom{n+j}{j}$$

b)

$$\sum_{j=0}^m \binom{m}{j} \binom{n}{r+j}$$

PROBLEM B2: Prove that in a finite poset the cardinality of a maximum chain is equal to the minimum number of disjoint antichains into which the poset can be partitioned.

PROBLEM B3: A graph is a small degree graph if its maximum degree is at most five. A graph  $G$  is degree uniform if, for any integers  $d$  and  $d'$ , if  $G$  contains at least one vertex of degree  $d$  and one vertex of degree  $d'$ , then  $G$  contains the same number of vertices of degree  $d$  as  $d'$ . For example, every regular graph is degree uniform as are  $P_4$  and  $K_{m,m,2m}$ . Determine all sets of positive integers  $S$  for which there exists a degree uniform, small degree maximal planar graph  $G$  of order at least 4 such that  $S$  is the set of vertex degrees of  $G$ .