Ph.D. Qualifying Examination in Algebra

Department of Mathematics
University of Louisville

Spring 2007

1 Groups

Do any two problems from this section.

1. Classify up to isomorphism all groups of order 175.

2. Let H and N be subgroups of a group G with N normal. Prove that $NH = HN$ and that this set is a subgroup of G.

3. Do two parts to receive full credit.

   (a) Let G be the multiplicative group of all nonsingular 2×2 matrices with rational numbers. Show that $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has order 4 and $h = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ has order 3, but that gh has infinite order.

   (b) Show that the additive group $H = (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$ contains nonzero elements a, b of infinite order such that $a + b$ is nonzero and has finite order.

4. Prove that every finitely generated subgroup of the additive group of rational numbers is cyclic.

2 Rings

Do two problems from this section. **One problem should include either Problem 1 or 2 (but not both)**.

1. The following is a well known fact: if K is a commutative ring with identity and I is an ideal of K, then K/I is a field if and only if I is a maximal ideal of K. Answer all the parts to receive full credit.

   (a) Find all the maximal ideals of $\mathbb{Z}$. (You may use the fact that every ideal of $\mathbb{Z}$ is of the form $n\mathbb{Z}$.)

   (b) Determine whether the ideal $(3, x)$ is a maximal ideal in $\mathbb{Z}[x]$.

   (c) Determine whether the ideal (x) is a maximal ideal in $\mathbb{Z}[x]$.

2. Do all the parts to receive full credit.

   (a) Define *prime ideal* and *maximal ideal* in a commutative ring R with identity.

   (b) Let R and S be commutative rings with identities 1_R and 1_S, respectively, and let $f : R \to S$ be a ring homomorphism such that $f(1_R) = 1_S$. If P is a prime ideal of S show that $f^{-1}(P)$ is a prime ideal of R.

   (c) Let f be as in part (b). If M is a maximal ideal of S, is $f^{-1}(M)$ a maximal ideal of R? Prove that it is or give a counter example.

3. Let

$$A = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + c = b + d, \quad a, b, c, d \in \mathbb{Z} \right\}.$$

   It is easy to see that A is a subring of $M_2(\mathbb{Z})$ (the ring of 2×2 matrices with elements from $\mathbb{Z}$). Do the two parts to receive full credit.

   (a) Let R be the ring of 2×2 lower triangular matrices $\begin{bmatrix} m & 0 \\ n & p \end{bmatrix}$ with elements from $\mathbb{Z}$. Consider the map $f : R \to A$ defined by

$$f\left(\begin{bmatrix} m & 0 \\ n & p \end{bmatrix}\right) = \begin{bmatrix} m - n & m - n - p \\ n & n + p \end{bmatrix}.$$

   Is f a homomorphism of rings? Justify the answer.

   (b) Are the rings R and A isomorphic? Explain.

4. Show that a proper ideal M in a commutative ring R is maximal if and only if for every $r \in R \setminus M$ there exists $x \in R$ such that $1 - rx \in M$.

3 Fields

Do any two problems from this section.

1. Prove that $\mathbb{Q}(\sqrt{3}+\sqrt{5}) = \mathbb{Q}(\sqrt{3}, \sqrt{5})$. Describe the lattice of subgroups of $\mathrm{Gal}(\mathbb{Q}(\sqrt{3}, \sqrt{5})/\mathbb{Q})$ and the lattice of subfields of $\mathbb{Q}(\sqrt{3}, \sqrt{5})$.

2. Do all the parts to receive full credit.

   (i) Show that $g = X^3 - 3X - 1$ is an irreducible polynomial over $\mathbb{Q}$.

   (ii) It is known that there is a simple extension field $\mathbb{Q}(u)$ of $\mathbb{Q}$ such that u is a root of g and $[\mathbb{Q}(u) : \mathbb{Q}] = 3$. How is $\mathbb{Q}(u)$ defined? List the elements of a basis of $\mathbb{Q}(u)$ over $\mathbb{Q}$.

   (iii) Show that there exists a splitting extension K of g with $[K : \mathbb{Q}] \leq 6$.

3. Prove that one of $2, 3$ or 6 is a square in the finite field $\mathbb{F}_p$ for every prime p. Conclude that the polynomial

$$x^6 - 11x^4 + 36x^2 - 36 = (x^2 - 2)(x^2 - 3)(x^2 - 6)$$

   has a root modulo p for every prime p but has no root in $\mathbb{Z}$.