

# ANALYSIS QUALIFIER

MAY 7, 2007

## 1. TRUE/FALSE

Determine whether the following statements are true or false and explain how you reached your conclusion. Each problem is worth 10 points. You receive 5 points for a correct answer and 5 additional points for a correct reason.

**Problem 1.** If

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

then  $f$  is Riemann integrable over any compact interval.

**Problem 2.** If  $f$  is continuous on  $[0, 1]$  and  $f'(x) \leq -\frac{1}{2}$  a.e., then  $f(0) > f(1)$ .

**Problem 3.** If  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  is a sequence of functions converging pointwise to  $f$ , then  $f_n$  converges in measure to  $f$ .

**Problem 4.** Let  $S$  be a countable and dense subset of  $[a, b]$ . There is a function  $f : [a, b] \rightarrow \mathbb{R}$  that is of bounded variation, continuous at each point of  $S$  and discontinuous everywhere else.

**Problem 5.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and let  $a \in \mathbb{R}$ . If  $f'(a) > 0$  then  $f$  is increasing on a neighborhood of  $a$ .

## 2. PROOFS

Each of the following problems is worth 25 points. At most three of your solutions will be graded. If you submit more than three solutions, please indicate which three should be graded.

**Problem 1.** If  $E \subset \mathbb{R}$  is a Lebesgue measurable set such that  $|E \cap I| \leq \frac{2}{3}|I|$  for every bounded interval  $I$ , then  $|E| = 0$ .

**Problem 2.** Let  $f_n$  and  $f$  be functions from  $L^1([0, 1])$  such that  $f_n \rightarrow f$  in  $L^1([0, 1])$ . There is a subsequence  $f_{n_j}$  of  $f_n$  converging to  $f$  a.e. on  $[0, 1]$ .

**Problem 3.** Let  $\{r_k\}$  be an enumeration of the rational numbers in  $\mathbb{R}$ . Define

$$A_{j,k} = \left( r_k - 2^{-(j+k)}, r_k + 2^{-(j+k)} \right), \quad A_j = \bigcup_{k=1}^{\infty} A_{j,k} \quad \text{and} \quad A = \bigcap_{j=1}^{\infty} A_j.$$

Prove that  $\mathbb{Q} \subset A$  and  $|A| = 0$ . Is  $A = \mathbb{Q}$ ?

**Problem 4.** If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $F \subset \mathbb{R}$  is an  $\mathbf{F}_\sigma$  set, then  $f(F)$  is measurable.

**Problem 5.** Let  $K \subset \mathbb{R}$  be compact,  $x_0 \in K$  and  $x_n$  be a sequence in  $K$ . If every convergent subsequence of  $x_n$  converges to  $x_0$ , then  $x_n \rightarrow x_0$ .

## 3. MORE PROOFS

Each of the following problems is worth 25 points. At most three of your solutions will be graded. If you submit more than three solutions, please indicate which three should be graded.

**Problem 1.** Let  $\mu, \mu_n$  be finite measures defined on a measure space  $(X, \mathcal{M})$ . Suppose that

$$\lim_{n \rightarrow \infty} \int_E d\mu_n = \int_E d\mu$$

for every set  $E \in \mathcal{M}$ . Prove that

$$\lim_{n \rightarrow \infty} \int_X f d\mu_n = \int_X f d\mu$$

for every bounded measurable function  $f : X \rightarrow \mathbb{R}$ .

**Problem 2.** Give an example of a signed measure  $\nu$  defined on  $\mathbb{R}$  with the property that both sets in the Hahn decomposition are dense in  $\mathbb{R}$ .

**Problem 3.** Let  $(X, \mathcal{M}, \mu)$  be a measure space and  $f_n : X \rightarrow \mathbb{R}$  be a sequence of measurable functions. The set  $C = \{x : f_n(x) \text{ converges}\} \in \mathcal{M}$ .

**Problem 4.** Let  $1 \leq p < \infty$ . Then  $L^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$ .

**Problem 5.** If  $f \in L^1(\mathbb{R})$ , then  $\lim_{n \rightarrow \infty} \frac{1}{2n} \int_{-n}^n f = 0$ .