Combinatorics Qualifying Exam

May 16, 2007

This examination consists of two parts, A and B. Part A contains **six** problems of which you must select four to do. Part B contains **three** problems of which you must select two to do. Each problem in part A is worth 15 points and each problem in part B is worth 20 points. Only hand-in your solutions to four problems from part A and two from part B. Please do not turn-in more solutions since only the first four solutions from part A will be graded and only the first two solutions from part B will be graded.

Begin each problem on a new sheet of paper and be sure to label each page of your work with the problem number and your name.

In each question, if you appeal to a theorem within your solution, you must carefully state the entire theorem. All graphs, unless otherwise stated, should be understood to be finite and simple.

PART A: **(15 points each)** *Do any four.* **Time: 2 hours.**

PROBLEM A1: A *strong tournament* is an orientation of a complete graph in which there is a directed path connecting any ordered pair of vertices.

a) Prove that every strong tournament T of order $n > 3$ has at least two vertices v such that $T - v$ is a strong tournament.

b) For each $n \geq 4$ give an example of a strong tournament with only two such vertices.

PROBLEM A2: You walk in the xy-plane according to the following rules:

R1) You start from the point $(0, 3)$ on the y-axis.

R2) In each step, your x-coordinate increases by 1.

R3) In each step, your y-coordinate increases by 1 or decreases by 1, each with probability $\frac{1}{2}$.

(a) What is the number of all possible walks ending at $(9, 3)$?

(b) What is the number of all possible walks ending at $(9, 2)$?

(c) Find the probability that after nine steps you will be higher than at the start.

(d) What is the number of all possible walks ending at $(9, 2)$ but not touching the x-axis?

(e) Find the probability that after nine steps you will be higher than at the start and you did not touch the x-axis.

PROBLEM A3:

a) Find an exponential generating function for the number h_n of ways to color the cells of a $1 \times n$ array with the colors red, white, and blue, where the number of red squares is even and there is at least one blue square.

b) Find a compact formula for h_n.

PROBLEM A4: Prove or disprove: If G is a connected, simple graph that does not contain P_4 or C_3 as an induced subgraph, then G is a complete bipartite graph.

PROBLEM A5: Consider the set S of all positive integer divisors of 144 (including 1 and 144) ordered by the relation $a \leq b$ if and only if a divides b.

a) Draw the Hasse diagram of this partial order.

b) Give an example of a longest chain and a largest antichain in this partial order.

c) Decompose S into a minimum number of antichains.

PROBLEM A6: Show that if a simple graph on n vertices has k components, then it has at most $\frac{1}{2}(n - k)(n - k + 1)$ edges.

PART B: **(20 points each)** *Do any two.* **Time: 1 hour and 20 minutes.**

PROBLEM B1: Evaluate the given sum. Justify your answer:

a)

$$\sum_{k=1}^{n} k^2 \binom{n}{k}$$

b)

$$\sum_{k=1}^{n} k \binom{n}{k}^2$$

PROBLEM B2:

a) Prove that if G is a graph with $q(p-1)+1$ vertices and minimum degree $q(p-2)+1$ then G contains K_p as a subgraph.

b) Prove that any coloring of the edges of $K_{q(p-1)+1}$ with colors red and blue contains either a red K_p or a blue $K_{1,q}$.

c) Exhibit a red-blue coloring of the edges of $K_{q(p-1)}$ with neither a red K_p nor a blue $K_{1,q}$.

PROBLEM B3: Let G be a simple graph with n vertices and independence number $r \geq 2$. Prove that if D is an acyclic orientation of G, then D contains a directed path of length at least $\lceil \frac{n}{r} \rceil - 1$.