Probablity and Mathematical Statistics Qualifier Exam

May 18, 2007

Remark : Choose three problems in 1-4, and choose three problems in 5-8

Name :

Problem 1: Let $(\Omega, \mathcal{F}, \mathbf{P}) = ([0,1), Borel, Lebesgue)$. For $n = 0, 1, 2, 3...$ let $Q_n = \sigma([\frac{k}{2^n}, \frac{k+1}{2^n}) : 0 \leq k < n)$ be the dyadic partitions of $[0,1)$. Let $X(s) = s, , 0 \leq s < 1$.
(a)Find the equation of and sketch $E(X|Q_0)$,$E(X|Q_1)$ and $E(X|Q_2)$.

(b)Let $Y_n = E(X|Q_n)$. Show that $E(Y_6|Q_4) = Y_4$. Find $E(Y_4|Q_6)$.

Problem 2:
(a) Suppose that $\Omega \in \mathcal{F}$ and that $A, B \in \mathcal{F}$ implies $A - B \in \mathcal{F}$. Show that $\mathcal{F}$ is a field.

(b) Suppose that $\Omega \in \mathcal{F}$ and that $\mathcal{F}$ is closed under the formation of complements and finite disjoint unions. Find a counterexample which shows that $\mathcal{F}$ need not be a field.

Problem 3:
(a) Let $X, Y \in L^1$. If X and Y are independent, show that $XY \in L^1$. Give an example to show XY need not be in L^1 in general.

(b) Let X_n be i.i.d. random variables with $P(X_n = 1) = \frac{1}{2}$ and $P(X_n = -1) = \frac{1}{2}$. Show that $\frac{1}{n}\sum_{j=1}^{n} X_j$ converges to 0 in probability.

Problem 4:
(a) Suppose that $X_1, X_2, \ldots$ is an independent sequence and Y is measurable $\sigma(X_n, X_{n+1}, \ldots)$ for each n. Show that there exists a constant a such that $P(Y = a) = 1$.

(b) Prove for integrable X that

$$E[X] = \int_0^\infty P(X > t)\ dt - \int_{-\infty}^0 P(X < t)\ dt.$$

Problem 5:
(a) f $Z \sim N(0,1)$, prove the following.

$$P(|Z| \geq t) \geq \sqrt{\frac{2}{\pi}} \frac{t}{t^2+1} e^{-\frac{t^2}{2}}.$$

(b) Let $X_1, X_2, ..., X_n$ be an i.i.d. sample from $\mathrm{B}(n,p)$. Show that there is no unbiased estimator of $g(p) = 1/p$.

(c) Suppose there are n AA Financial bonds, and let X_i denote the spread return, in a given month, of the ith bond, $i = 1, \ldots, n$. Suppose that they are all independent and follows normal distribution with a common mean μ but different variances σ_i^2. Show that the joint distribution of $X = (X_1, X_2, \ldots, X_n)$ makes an exponential family and find $C(x), h(x), w_i(x)$ and $T_i(x)$.

Problem 6: Let $X_1, X_2, ..., X_n$ be i.i.d. random sample with density

$$f(x|\mu,\sigma) = \frac{1}{\sigma} \exp(-\frac{x-\mu}{\sigma}) I_{(\mu,\infty)}(x),$$

where $\sigma > 0$.
(a) Find a two-dimensional sufficient statistic for (μ, σ).

(b) Find the distribution of $\frac{n}{\sigma}(X_{(1)} - \mu)$.

(c) Find the distribution of $\sum_{i=1}^n (X_i - X_{(1)})/\sigma$.

(d) Using (2) and (3), check that the sufficient statistic you found in (a) is also complete.

Problem 7:

(a) Let $X_1, X_2, ..., X_n$ be i.i.d. from Uniform$(0, \theta)$. Find the Complete Sufficient Statisitcs of θ. Also, find the UMVUE of θ.

(b) Let $X_1, X_2, ..., X_n$ be i.i.d. from Poisson(λ). Find a Complete Sufficient Statistic of λ. Also, find the UMVUE of $\eta = P(X_1 = a)$.

Problem 8:

(a) Let $X_1, X_2, ..., X_n$ be i.i.d. from a distribution having p.d.f. of the form $f(x) = \theta x^{\theta-1} I_{(0,1)}(x)$. Find the Rejection Region of the most poweful test for $H_0 : \theta = 1$ versus $H_1 : \theta = 2$

(b) Let $X_1, X_2, ..., X_n$ be i.i.d. from a distribution having p.d.f. of the form $f(x) = e^{-(x-\theta)} I_{[\theta,\infty)}(x)$. Find the likelihood ratio test for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.