

**Mathematical Modeling Qualifier Exam**  
**May, 2007**

Name \_\_\_\_\_ ID \_\_\_\_\_

There are two parts in this Exam.

Do any 3 problems out of the 4 problems in Part I, and do any 3 problems out of the 4 problems in Part II.

You must show all work to receive full credit. If you do more problems than what are required, you MUST indicate what problems you want to be graded. Otherwise, they may be randomly chosen to grade.

**Part I:**

1. (a) If the initial energy is sufficiently large, determine an expression for the time it takes a pendulum to go completely around.  
(b) Estimate the time if the energy is very large ( $E \gg 2g$ ).

2. Assume that a mass  $m$  satisfies  $m \frac{d^2 x}{dt^2} = -(x^2 - 1)$ .

- (a) What are the equilibrium positions?
- (b) Derive an expression for conservation of energy.
- (c) Sketch trajectories in phase plane for the equation.

3. Consider the traffic model

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho} \frac{\partial \rho}{\partial x} = 0,$$

with the initial condition

$$\rho(x, 0) = \begin{cases} \frac{1}{2} & x \leq 0 \\ \frac{3}{2} & 0 < x < 2 \\ 1 & x \geq 2. \end{cases}$$

Suppose that the traffic flux is given by the function  $q(\rho) = 3\rho(2 - \rho)$ . Sketch the characteristics and shock. Find formulas for the density  $\rho(x, t)$  and the shock  $x_s(t)$ .

4. Assume that  $u = u_{max}(1 - \rho/\rho_{max})$ .

- (a) Show that the time of intersection of neighboring characteristics starting at  $x(0) = x_1$  is

$$t = \frac{\rho_{max}}{2u_{max} \frac{d\rho}{dx_1}}.$$

- (b) If at  $t = 0$ ,

$$\rho(x, 0) = \rho_{max} \exp\left(\frac{-x^2}{L^2}\right).$$

- (1) Sketch the initial density. (2) Determine the time of the first shock. (3) Where does this shock first occur?

## Part II:

1. Consider a model of two competing species

$$x' = x(1 - x - y), \quad y' = y(2 - x - \beta y),$$

where the parameter  $\beta$  satisfies the condition  $0 < \beta < 2$ .

- (a) Find all critical points and their local stability.
  - (b) Show that the point  $(0, \frac{2}{\beta})$  is a global attractor for all positive solutions.
2. Consider the system

$$\begin{aligned}x' &= x - y - x^3, \\y' &= x + y - y^3.\end{aligned}$$

- (a) This system has only one critical point. Determine its local stability.
  - (b) Show that this system has a limit cycle.
  - (c) Show that this system has at most one limit cycle.
3. Consider the following one-parameter system of differential equations in polar form:

$$\dot{r} = r(\mu - r)(\mu - 2r), \quad \dot{\theta} = -1.$$

Plot phase portraits for  $\mu < 0$ ,  $\mu = 0$  and  $\mu > 0$ . Sketch corresponding bifurcation diagrams.

4. (a) Solve the following differential equations

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1.$$

- (b) By considering the line segment  $\Sigma = \{(x, y) \in \mathbb{R}^2 : 0 \leq x < \infty\}$ , find the Poincaré map for this system.