Ph.D. Qualifying Examination in Algebra

Department of Mathematics
University of Louisville
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1 Groups

Do any two problems from this section.

1. Show that a group of order $1000 = 2^4 \cdot 5^4$ cannot be simple.

2. Let K be a normal subgroup of a group G and suppose that K has exactly two elements. Show that K is contained in the center $Z(G)$ of G.

3. Prove that if H is a normal subgroup of a group G of prime index p, then for all $K \leq G$ either

(i) $K \leq H$ or

(ii) $G = HK$ and $|K : K \cap H| = p$.

2 Rings

Do any two problems from this section.

1. Let $R = C[0,1]$ be the set of all continuous real-valued functions on $[0,1]$. Define addition and multiplication on R as follows. For $f, g \in R$ and $x \in [0,1]$,

$$(f+g)(x) = f(x) + g(x)$$
$$(fg)(x) = f(x)g(x)$$

a) Show that R with these operations is a commutative ring with identity.

b) Find the units of R.

c) If $f \in R$ and $f^2 = f$, then show that $f = 0_R$ or $f = 1_R$.

2. In the ring $\mathbb{Z}$ of integers the following conditions on a nonzero ideal I are equivalent: (i) I is prime; (ii) I is maximal; (iii) $I = (p)$ with p prime.

3. Consider the ring $\mathbb{Z}[i]$ of Gaussian integers.

(a) Show that any prime $p \in \mathbb{Z}$ with $p \equiv 3 \pmod 4$ is irreducible over $\mathbb{Z}[i]$.

(b) Prove that $f(x) = x^3 + 12x^2 + 18x + 6$ is irreducible over $\mathbb{Z}$ and over $\mathbb{Z}[i]$.

3 Fields

Do any two problems from this section.

1. Let F be a field. Prove that if $[F(\alpha) : F]$ is odd then $F(\alpha) = F(\alpha^2)$.

2. Let $f(x) = x^4 - 5x^2 + 6$ over $\mathbb{Q}$.
(a) Find the splitting field K of $f(x)$ over $\mathbb{Q}$.
(b) Find the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$.
(c) Draw the diagram of the subgroups of G and the diagram of corresponding fixed fields.

3. Let F be a finite field with multiplicative identity 1_F. Prove that when you multiply all the nonzero elements of F, then the answer is -1_F. (Here -1_F is the additive inverse of 1_F.)