Combinatorics Qualifying Exam

August, 2008

This examination consists of two parts, A and B. Part A contains **six** problems of which you must select four to do. Part B contains **three** problems of which you must select two to do. Each problem in part A is worth 15 points and each problem in part B is worth 20 points. Only hand-in your solutions to four problems from part A and two from part B. Please do not turn-in more solutions since only the first four solutions from part A will be graded and only the first two solutions from part B will be graded.

Begin each problem on a new sheet of paper and be sure to label each page of your work with the problem number and your name.

In each question, if you appeal to a theorem within your solution, you must carefully state the entire theorem. All graphs, unless otherwise stated, should be understood to be finite and simple.

Part A: **(15 points each)** *Do any four.* **Time: 2 hours.**

Problem A1: A *tournament* is a complete graph in which every edges has been given an orientation. Prove that every tournament has a directed Hamiltonian path.

Problem A2: Solve the recurrence

$$a_n = 5a_{n-1} - 6a_{n-2} \qquad (\text{ for } n \geq 2),$$

with initial conditions $a_0 = 1$ and $a_1 = 1$.

Problem A3: Suppose that G is a connected planar graph that can be drawn in the plane so that all faces have an even number edges on their boundary. Prove that the vertices of G can be properly 2-colored.

Problem A4: All points of the plane that have integer coordinates are colored so that each such point receives one of the three colors: red, blue or green. Prove that there must be a rectangle whose four corner vertices are all of the same color.

Problem A5: Prove that if every chain and every antichain of a poset P is finite, then P is finite.

Problem A6: Let G be a graph in which any two odd cycles intersect.

a) Prove that G is 5-colorable.

b) Give an example to show that 4 colors do not suffice.

PART B: **(20 points each)** *Do any two.* **Time: 1 hour and 20 minutes.**

PROBLEM B1:

a) Find, with a proof, the number of edges in the extremal graph on 6 vertices without K_4 as a subgraph.

b) Find, with a proof, the number of edges in the extremal graph on 6 vertices without C_4 as a subgraph.

PROBLEM B2: Prove the given identity:

a)

$$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}$$

b)

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

PROBLEM B3: Prove or disprove: If G is a connected, simple graph that does not contain P_4 or C_3 as an induced subgraph, then G is a complete bipartite graph.