

Mathematical Modeling Qualifier Exam
Aug, 2008

Name _____ ID _____

There are two parts in this Exam.

Do any 3 problems out of the 4 problems in Part I, and do any 3 problems out of the 4 problems in Part II.

You must show all work to receive full credit. If you do more problems than what are required, you MUST indicate what problems you want to be graded. Otherwise, they may be randomly chosen to grade.

Part I:

1. Consider

$$m \frac{d^2x}{dt^2} = -c \frac{dx}{dt} + \alpha(e^{\beta x} - 1),$$

where α, β and c are positive constants.

- (a) Find equilibrium point of the system, and determine its linear stability.
 - (b) Derive a first-order differential equation describing the phase plane ($\frac{dx}{dt}$ as a function of x).
 - (c) Find some typical isoclines for the case $\alpha = \beta = m = c = 1$. Use the isoclines to roughly sketch solutions in the phase plane based on the first-order differential equation. Show details.
2. Consider a nonlinear pendulum

$$L \frac{d^2\theta}{dt^2} = -g \sin \theta.$$

- (a) Derive the energy integral
- $$\frac{L}{2} \left(\frac{d\theta}{dt} \right)^2 = g(\cos \theta - 1) + E.$$
- (b) Use the energy integral to obtain an expression for the period of oscillation.
 - (c) Estimate the period if the energy is very large ($E \gg 2g$). (Hint: $2g/E \ll 1$.)
3. (Green light problem) Assume that a traffic is stopped by an extremely long red light. Suppose that at $t = 0$ the traffic light turns from red to green. The traffic flow problem for the case may be modeled as following:

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho} \frac{\partial \rho}{\partial x} = 0,$$

with the initial condition

$$\rho(x, 0) = \begin{cases} \rho_{max} & x \leq 0 \\ 0 & x > 0, \end{cases}$$

where ρ is the traffic density and q the traffic flux, $q = \rho u(\rho) = u_{max} \rho (1 - \rho/\rho_{max})$. If $u_{max} = 40$ and $\rho_{max} = 225$,

- (1) Solve $\rho(x, t)$ for $x \in R, t > 0$.
- (2) Find the car trajectory $x(t)$ for the car at $x(0) = -1$. When will the car pass the traffic light at $x = 0$?
- (3) Sketch the graph of characteristics, the density at $t = 1$, and the car trajectory.

4. (Red light problem) Assume that a traffic is moving at a constant density ρ_0 , and then stopped at a point $x = 0$ by a red light for a finite amount of time T_1 . The traffic light turns green again at $t = T_1$. The traffic flow problem for the case may be modeled as following:

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho} \frac{\partial \rho}{\partial x} = 0,$$

with the initial condition

$$\rho(x, 0) = \begin{cases} \rho_0 & x < 0 \\ \rho_{max} & x = 0^- \\ 0 & x = 0^+ \\ \rho_0 & x > 0, \end{cases}$$

where ρ is the traffic density and q the traffic flux, $q = \rho u(\rho) = u_{max} \rho (1 - \rho / \rho_{max})$. If $u_{max} = 40$, $\rho_{max} = 200$, $\rho_0 = 100$, and $T_1 = 2$.

- (1) Find the equations of the shock before the red light ('left' shock) and the shock after the red light ('right' shock).
- (2) Solve $\rho(x, t)$ for $x \in R$, $0 < t < T_1$.
- (3) Assume that the lead car catches up to the uniformly moving traffic at $t = T_u$, and the line of stopped cars completely dissipates at $t = T_d$. Solve $\rho(x, t)$ for $x \in R$, $T_1 < t < T_2$, where $T_2 = \min\{T_u, T_d\}$.
- (4) Sketch the graph of characteristics, the shocks and fan-like solutions. Sketch the graphs of $\rho(x, t)$ for $t = 0$, $t = T_1$, and $t = T_2$.

Part II:

1. A certain species of fish can be divided into three age groups, each one year long. The Leslie matrix for the female portion of the population is given by

$$L = \begin{pmatrix} 0 & 4 & 8 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{pmatrix}$$

- (a) Find the long-term growth rate and the long-term distribution of age classes of the population. (**Hint:** The equation $-\lambda^3 + 2\lambda + 1 = 0$ has a root $\lambda = -1$.)
- (b) Determine a sustainable harvesting policy, if harvest the oldest age class only. (**Hint:** $(1 - d_1)[b_1 + b_2c_1(1 - d_2) + b_3c_1c_2(1 - d_2)(1 - d_3)] = 1$.)
- (c) Describe how you determine the long-term distribution of the age classes of the population if the sustainable harvesting policy in (b) is applied. (**Note:** You are not required to find the eigenvalues, give only formulas and describe the procedure.)
2. Consider the following one-parameter system of differential equations in polar form:

$$\dot{r} = r(\mu^2 - r^2), \quad \dot{\theta} = 1.$$

Plot phase portraits for $\mu < 0$, $\mu = 0$ and $\mu > 0$. Sketch corresponding bifurcation diagrams.

3. Consider the Hamiltonian system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 \end{aligned}$$

- (a) Find all critical points and determine their local stability
- (b) Find the Hamiltonian of the system and sketch a phase portrait.
4. Solve the following differential equations

$$\dot{r} = r(4 - r^2), \quad \dot{\theta} = 1.$$

By considering the line segment $\Sigma = \{(x, y) \in \mathbb{R}^2 : 0 \leq x < \infty\}$, find the Poincaré map for this system.