Ph.D. Qualifying Examination in Algebra

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1 Groups

Do any two problems from this section.

1. Show that if G is a cyclic group then every subgroup of G is cyclic.

2. Let G be a group and $Z(G)$ be its center. Prove that if $G/Z(G)$ is cyclic then G is Abelian.

3. Let H and N be subgroups of a group G with N normal. Prove that $NH = HN$ and that this set is a subgroup of G. (Here $NH = \{nh | n \in N, h \in H\}$, $HN = \{hn | h \in H, n \in N\}$.)

2 Rings

Do any two problems from this section.

1. Let $\mathbb{Z}$ be the ring of integers. Prove that the polynomial ring $\mathbb{Z}[x]$ is not a principal ideal domain. Is it a unique factorization domain?

2. Prove that every nonzero prime ideal in a principal ideal domain is a maximal ideal.

3. Show that a proper ideal M in a commutative ring R is maximal if and only if for every $r \in R \setminus M$ there exists $x \in R$ such that $1 - rx \in M$.

3 Fields

Do any two problems from this section.

1. If F is a finite dimensional extension field of K, then prove that F is finitely generated and algebraic over K.

2. Let $f(x) = x^4 - 2$ over $\mathbb{Q}$.
(a) Find the splitting field K of $f(x)$ over $\mathbb{Q}$.
(b) Find the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$.
(c) Draw the diagram of the subgroups of G and the diagram of corresponding fixed fields.

3. Show that the polynomial $g(x) = x^2 + x + 1 \in \mathbb{Z}_2[x]$ is irreducible in $\mathbb{Z}_2[x]$. Using this polynomial construct a field of four elements. Construct the addition and the multiplication table for this field.