Combinatorics Qualifying Exam

May 9, 2008

This examination consists of two parts, A and B. Part A contains **six** problems of which you must select four to do. Part B contains **three** problems of which you must select two to do. Each problem in part A is worth 15 points and each problem in part B is worth 20 points. Only hand-in your solutions to four problems from part A and two from part B. Please do not turn-in more solutions since only the first four solutions from part A will be graded and only the first two solutions from part B will be graded.

Begin each problem on a new sheet of paper and be sure to label each page of your work with the problem number and your name.

In each question, if you appeal to a theorem within your solution, you must carefully state the entire theorem. All graphs, unless otherwise stated, should be understood to be finite and simple.

PART A: **(15 points each)** *Do any four.* **Time: 2 hours.**

PROBLEM A1:

a) Show that if G is a 2-connected graph containing a vertex that is adjacent to at least three vertices of degree 2, then G is not Hamiltonian.

b) The subdivision graph $S(G)$ of a graph G is the graph obtained from G by replacing each edge uv by a vertex w and edges uw and vw. Determine, with a proof, all graphs G for which $S(G)$ is Hamiltonian.

PROBLEM A2: Recall that a set S of vertices of a graph G is independent if every two vertices of S are not adjacent in G. The independence number, $\beta(G)$, of a graph G is the maximum cardinality among independent sets of vertices of G. Prove that a graph G is bipartite if and only if $\beta(H) \geq \frac{|V(H)|}{2}$, for every subgraph H of G.

PROBLEM A3: A simple planar graph $G(V, E)$ has only vertices of degree $3, 4, 5$, and 6, with the same number of each type. Find the order $|V|$ and the size $|E|$ of G.

PROBLEM A4: All points of the plane that have integer coordinates are colored so that each such point receives one of the three colors: red, blue or green. Prove that there must be a rectangle whose four corner vertices are all of the same color.

PROBLEM A5: Solve the recurrence relation

$$a_0 = 1 \qquad \text{and} \qquad a_n = 3\sum_{i=0}^{n-1} a_i, \qquad \text{for all } n \geq 1.$$

PROBLEM A6: A function $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is *monotone* if $x < y$ implies $f(x) \leq f(y)$. Determine the number of such monotone functions.

Part B: **(20 points each)** *Do any two.* **Time: 1 hour and 20 minutes.**

Problem B1:

a) Find, with a proof, the number of edges in the extremal graph on 6 vertices without K_4 as a subgraph.

b) Find, with a proof, the number of edges in the extremal graph on 6 vertices without C_4 as a subgraph.

Problem B2: Consider the poset $(P, \leq)$ where P is the set of all subsets of $\{1,2,3,4,5,6,7\}$ with odd cardinality and $\leq$ is the inclusion relation.

a) Find the number of elements in P.

b) Find all minimal and maximal elements of $(P, \leq)$.

c) Determine the length ℓ and the width w of $(P, \leq)$.

d) Give an example of a chain of cardinality ℓ and an antichain of cardinality w.

Problem B3: The automorphism group $Aut(G)$ of the graph G shown below consists of four permutations.

a) List all elements of $Aut(G)$ using cycle notation.

b) Find the cycle index of the group $Aut(G)$.

c) Find the number of different 3-colorings of the vertices of G.

d) Find the number of different labelings $\ell(G)$ of the vertices of the graph G if the available labels are a, b, c, d, e, f.