Mathematical Modeling Qualifier Exam

May, 2008

Name _____ **ID** _____

There are two parts in this Exam.

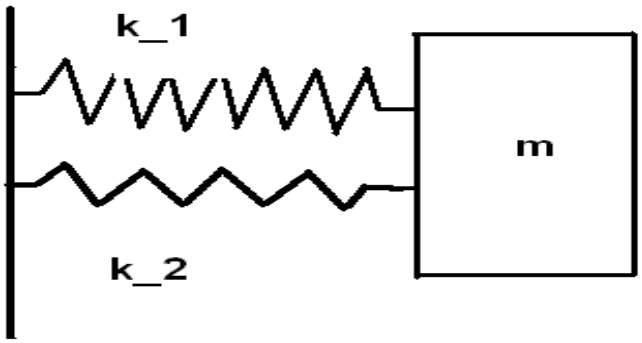
Do any 3 problems out of the 4 problems in Part I, and do any 3 problems out of the 4 problems in Part II.

You must show all work to receive full credit. If you do more problems than what are required, you MUST indicate what problems you want to be graded. Otherwise, they may be randomly chosen to grade.

Part I:

1. Suppose that a mass m were attached to two springs in parallel (see figure 1).

Figure 1:



Set up the model for the spring-mass system and answer the following questions.

(a) What position of the mass would be called the equilibrium position of the mass.

(b) Show that the mass executes simple harmonic motion about its equilibrium position.

(c) What is the period of oscillation?

2. Consider a nonlinear pendulum with Newtonian damping

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta - \beta\frac{d\theta}{dt}\left|\frac{d\theta}{dt}\right|,$$

where $\beta > 0$.

(a) By introducing the phase plane variable $v = \frac{d\theta}{dt}$, show that

$$\frac{dv}{d\theta} = \frac{-g \sin\theta - \beta v|v|}{Lv}.$$

(b) Instead of sketching the isoclines, show that

$$L\frac{d}{d\theta}(v^2) \pm \beta v^2 = -g \sin\theta.$$

(c) Under what conditions does the + or − sign apply?

(d) This is a linear differential equation for v^2. Solve this equation.

(e) Using this solution, roughly sketch the phase plane (trajectories on the $\theta - v$ plane).

3. Consider the traffic model

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = 0,$$

with the initial condition

$$\rho(x,0) = \begin{cases} 2 & x \le 0 \\ 0 & 0 < x < 4 \\ 4 & x \ge 4. \end{cases}$$

Suppose that the traffic flux is given by the function $q(\rho) = 12\rho(1 - \frac{\rho}{4})$. Sketch the characteristics and the shocks. Find formulas for the density $\rho(x,t)$ and the shock $x_s(t)$.

4. The traffic flow in a highway with entrances and exits may be modeled as

$$\frac{\partial \rho}{\partial t} + \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} = \beta,$$

where

$$\beta(x,t) = \begin{cases} 0 & x \le 0 \\ 2 & x > 0. \end{cases}$$

Suppose that $q(\rho) = 4\rho(2 - \rho)$ and the initial density is

$$\rho(x,0) = \begin{cases} 2 & x \le 0 \\ 0 & x > 0. \end{cases}$$

(a) Sketch the graphs of characteristics and typical densities for different time t.

(b) Find formulas for the density $\rho(x,t)$ and the characteristics.

Part II:

1. Consider

$$x' = -\frac{1}{2}y(1+x) + x(1-4x^2-y^2), \;\; y' = 2x(1+x) + y(1-4x^2-y^2).$$

(a) Prove that the origin is a unique critical point of the system and determine stability of the origin.

(b) Use the Lyapunov function $V(x,y) = (1-4x^2-y^2)^2$ to show that $\Lambda^+(p) = \{(x,y)|4x^2+y^2 = 1\}$ for each $p \in \mathbf{R}^2$. (That is, $\{(x,y)|4x^2+y^2=1\}$ is the global attractor of the system.)

2. (a) Prove that the system

$$x' = x(1-4x+y), \;\; y' = y(2+3x-2y)$$

has no limit cycles by applying Bendixson's criteria with $\psi = x^m y^n$.

(b) Consider the Hamiltonian system

$$\dot{x} = y,$$
$$\dot{y} = x - x^3$$

Find the Hamiltonian of the system and sketch a phase portrait.

3. Consider the predator-prey system

$$x' = 2x - x^2 - xy,$$
$$y' = -y - y^2 + xy.$$

(a) Find all critical points.

(b) Discuss the linearized system in a neighborhood of each critical point. (Describe its type and stability.)

(c) Use isoclines to sketch a phase portrait for the nonlinear system. (Note: $x \geq 0, y \geq 0$. There is no limit cycle.)

(d) Interpret the solution in terms of species behavior.

4. Consider the one-parameter system of differential equations

$$x' = x^2 - x\mu^2,$$
$$y' = -y.$$

Find the critical points, plot phase portraits, and sketch a bifurcation diagram.