Multiple View Geometry in Computer Vision

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Camera Models
Lecture 9
In our last lecture, we model various types of cameras using a $3 \times 4$ matrix $\mathbf{P}$.

In this lecture, we will examine various pieces of this camera matrix $\mathbf{P}$. 
Null space of a matrix

Let \( \mathbb{R}^n \) be the set of \( n \)-tuples \( (x_1, x_2, ..., x_n)^T \) where \( x_1, x_2, ..., x_n \) are elements in \( \mathbb{R} \). Then \( \mathbb{R}^n \) is a vector space over \( \mathbb{R} \).

Given a matrix \( A \), by **null space** of \( A \), we mean the set

\[
N = \left\{ x \in \mathbb{R}^n \mid Ax = 0 \right\}.
\]

The null set \( N \) is a vector subspace of \( \mathbb{R}^n \).
General Projective Cameras

Recall that a camera is called a general projective camera if it can be represented by an arbitrary homogeneous $3 \times 4$ matrix $\mathbf{P}$ of rank 3.

A general projective camera $\mathbf{P}$ maps world points $\mathbf{X}$ to image points $\mathbf{x}$ according to $\mathbf{x} = \mathbf{P} \mathbf{X}$. 
The $3 \times 4$ matrix $P$ can be decomposed into blocks as

$$P = \begin{bmatrix} M & p_4 \end{bmatrix}$$

where $M$ is a $3 \times 3$ matrix and $p_4$ is the $4^{th}$ column.
Camera Center

The right null space of $P$

Since the $3 \times 4$ matrix $P$ has rank 3, it has a right null space. Suppose the right null space of $P$ is generated by the 4-vector $C$. Then $PC = 0$.

We want to show that this 4-vector $C$ is the camera center of the general projective camera $P$. 
Consider the line containing $C$ and any other point $A$ in 3-space. Points on this line can be represented by the convex combinations of $C$ and $A$, that is

$$X(\lambda) = \lambda A + (1 - \lambda) C$$

where $\lambda \in [0, 1]$. Since $PC = 0$, under the mapping $x = PX$ points on this line are projected to

$$x = PX(\lambda) = \lambda PA + (1 - \lambda) PC = \lambda PA.$$
• This last equation shows that all points on the line containing $C$ and $A$ are mapped to the same image point $PA$.

• This means that the line must be a ray through the camera center.

• Since for all choices of $A$ the line $X(\lambda)$ is a ray through the camera center, therefore $C$ is the homogeneous representation of the camera center.
Column Vectors

The vanishing points of the coordinates axes

Let $P_i$ be the $i^{th}$ column of the projective camera $P$ for $i = 1, 2, 3, 4$ (see figure below).

$$P = \begin{pmatrix} P_1 & P_2 & P_3 & P_4 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$
Let \((1, 0, 0, 0)^T\) be the point at infinity (or direction) along the \(x\)-axis in the world coordinates system (WCS). The image of the point \((1, 0, 0, 0)^T\) is given by

\[
\begin{pmatrix}
p_{11} & p_{12} & p_{13} & p_{14} \\
p_{21} & p_{22} & p_{23} & p_{24} \\
p_{31} & p_{32} & p_{33} & p_{34}
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
p_{11} \\
p_{21}
\end{pmatrix}
= P_1.
\]

Hence \(P_1\) is the vanishing point of the \(x\)-axis of the world coordinates system.
Similarly, \( P_2 \) and \( P_3 \) are the vanishing points of the \( y \)-axis and \( z \)-axis of the world coordinates system.
Row Vectors

The principal plane and coordinates axis planes

Let $P_i^T$ be the $i^{th}$ row vector of the projective camera $P$ for $i = 1, 2, 3$. That is

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{pmatrix} = \begin{pmatrix} P_1^T \\ P_2^T \\ P_3^T \end{pmatrix}.$$
Principal Plane

The principal plane consists of a set of points $\mathbf{X}$ whose images are at the line of infinity of the image plane.

Therefore

$$\mathbf{P} \mathbf{X} = (x, y, 0)^T \iff \mathbf{P}^3^\top \mathbf{X} = 0.$$
The equation $\mathbf{P}^3 \mathbf{T} \mathbf{X} = 0$ is an equation of a plane, and it implies that $\mathbf{P}^3$ is a 4-vector representing the principal plane.

If $\mathbf{C}$ is the camera center, then $\mathbf{P} \mathbf{C} = 0$. Therefore, in particular $\mathbf{P}^3 \mathbf{T} \mathbf{C} = 0$, and hence $\mathbf{C}$ lies on the principal plane $\mathbf{P}^3$. 

Plane defined by the third row $P^3T$ of the projection matrix
Axis Planes

The 4-vector $P^1$ represents a plane. If $X$ is a point in the plane $P^1$, then $P^1^T X = 0$. Hence

$$PX = \begin{pmatrix} P^1^T \\ P^2^T \\ P^3^T \end{pmatrix} \begin{pmatrix} X \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ w \end{pmatrix},$$

where $y = P^2^T X$ and $w = P^3^T X$. 
If $C$ is the camera center, then we have $PC = 0$. Hence in particular $P^1^T C = 0$. Therefore $C$ lies on the plane $P^1$.

- The points $X$ of the plane $P^1$ are mapped on to the $y$-axis of the image plane by the projection map $P$. Further, $C$ lies on the plane $P^1$. Hence the plane $P^1$ must be the $y$-axis plane.
Plane defined by the first row $P^1_T$
of the projection matrix

The plane $P^1$ is the $y$-axis plane.
Following a similar argument, one can show that the plane $\mathbb{P}^2$ must be the $x$-axis plane.
Plane defined by the second row $p^{2T}$ of the projection matrix

The plane $P^2$ is the $x$-axis plane.
Unlike the principal plane $P^3$, the axis planes $P^1$ and $P^2$ are dependent on the image $x$- and $y$-axes.

Shifting image origin shifts the $x$, $y$ axis planes!
Since the camera center $C$ lies on all three planes, and since these planes are distinct (as the $P$ has rank 3) it must lie on their intersection.
Planes defined by the rows of the projection matrix
Principal Point

- The point where the principal axis meets the image plane is called the principal point.
The row 4-vector $\mathbf{P}^3 = (p_{31}, p_{32}, p_{33}, p_{34})^T$ represents the principal plane. The vector $(p_{31}, p_{32}, p_{33})^T$ is the normal to the principal plane $\mathbf{P}^3$.

The 4-vector $\hat{\mathbf{p}}^3 = (p_{31}, p_{32}, p_{33}, 0)^T$ is a point at the infinity along the direction of this normal vector.

The principal point $x_0$ can be obtained by projecting the point at infinity $(p_{31}, p_{32}, p_{33}, 0)^T$ using the central projection mapping $\mathbf{P}$. 
That is, the principal point $x_0$ can be computed as

$$x_0 = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \end{pmatrix} \begin{pmatrix} p_{31} \\ p_{32} \\ p_{33} \\ 0 \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \begin{pmatrix} p_{31} \\ p_{32} \\ p_{33} \end{pmatrix}$$

$$= M m^3,$$

where $M$ is the left hand $3 \times 3$ submatrix of $P$ and $m^3^T$ is the third row of $M$. 

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The submatrix $M$ of $P$

$M$ is the left hand $3 \times 3$ submatrix of $P$ and $m^{3T}$ is the third row of $M$
Computation of Principal Point

\[ \hat{p}^3 = (p_{31}, p_{32}, p_{33}, 0) \]

\[ x_0 = P \hat{p}^3 = M m^3 \]

Principal point \( x_0 \) is the image of the point at infinity along \( z \)-axis.
The principal axis vector

The vector defining the front side of camera

• \( \mathbf{v} = \text{det}(\mathbf{M}) \mathbf{m}^3 \) is a vector in the direction of the principal axis and directed towards the front of the camera.

\[
\mathbf{v} = \text{det}(\mathbf{M}) \mathbf{m}^3 = (0, 0, 1)^T
\]
Action of a projective camera on points

Forward Projection: A general projective camera maps a point $X$ in space to an image point $x$ according to the rule $x = PX$. The points at infinity $D = (d^T, 0)^T$ map to

$$x = PD = [M | p_4]D = Md$$

and thus are only affected by $M$. 
**Back-projection:** Given an image point $x$ we want to determine the set of points that map to $x$.

This set is a ray in space that passes through the camera center.
Consider two points \( C \) and \( P^+x \), where \( P^+ \) is the pseudo-inverse of \( P \).

\( P^+ \) is pseudo-inverse of \( P \) means \( P^+ = P^T (PP^T)^{-1} \).

Thus the ray is the line formed by joining these two points

\[ X(\lambda) = P^+x + \lambda C. \]
Camera Depth

Given a point $X_0$ on the world space and the camera matrix $P$, we want to find the camera depth $z_0$. 
Let

\[ X_0 = (x_w, y_w, z_w, t_w)^T \]

be a point on the world space, and

\[ PX_0 = x_0 = (x_c, y_c, 1)^T w_c. \]

Then the signed depth \( z_0 \) is given by

\[ z_0 = \frac{w_c \text{ sign}(\det M)}{t_w \|m^3\|}. \]
END