Everything should be made as simple as possible, but no simpler.
— Albert Einstein

Direction: This homework worths 100 points and is due on November 28, 2003. In order to receive full credit, answer each problem completely and must show all work. Graduate students should do all problems in groups A and B; and the undergraduate students are required to do any 7 problems from the group A and any 8 problems from the group B. Group C is for extra credit; each problem worths 3 points.

GROUP A

1. Let \( X_1, X_2, \ldots, X_n \) be a random sample from a population with gamma density function

\[
f(x; \theta, \alpha) = \begin{cases} 
\frac{1}{\Gamma(\beta) \theta^\beta} x^{\beta-1} e^{-\frac{x}{\theta}} & \text{for } 0 < x < \infty \\
0 & \text{otherwise},
\end{cases}
\]

where \( \theta \) is an unknown parameter and \( \beta > 0 \) is a known parameter. Show that

\[
\left[ \frac{2 \sum_{i=1}^{n} X_i}{\chi^2_{1 - \alpha/2}(2n\beta)}, \frac{2 \sum_{i=1}^{n} X_i}{\chi^2_{\alpha/2}(2n\beta)} \right]
\]

is a \((1 - \alpha)\)100% confidence interval for the parameter \( \theta \).

2. Let \( X_1, X_2, \ldots, X_n \) be a random sample from a population with Weibull density function

\[
f(x; \theta, \alpha) = \begin{cases} 
\beta \left[ \frac{x}{\theta} \right]^{\beta-1} e^{-\frac{x}{\theta} x^\beta} & \text{for } 0 < x < \infty \\
0 & \text{otherwise},
\end{cases}
\]

where \( \theta \) is an unknown parameter and \( \beta > 0 \) is a known parameter. Show that

\[
\left[ \frac{2 \sum_{i=1}^{n} X_i^\beta}{\chi^2_{1 - \alpha/2}(2n)}, \frac{2 \sum_{i=1}^{n} X_i^\beta}{\chi^2_{\alpha/2}(2n)} \right]
\]
is a \((1 - \alpha)100\%\) confidence interval for the parameter \(\theta\).

3. Let \(X_1, X_2, ..., X_n\) be a random sample from a population with Pareto density function

\[
f(x; \theta, \alpha) = \begin{cases} \theta \beta^\theta x^{-(\theta + 1)} & \text{for } \beta \leq x < \infty \\ 0 & \text{otherwise,} \end{cases}
\]
where \(\theta\) is an unknown parameter and \(\beta > 0\) is a known parameter. Show that

\[
\left[ \frac{2\sum_{i=1}^{n} \ln \left( \frac{X_i}{\beta} \right)}{\chi^2_{1 - \frac{\alpha}{2}}(2n)}, \frac{2\sum_{i=1}^{n} \ln \left( \frac{X_i}{\beta} \right)}{\chi^2_{\frac{\alpha}{2}}(2n)} \right]
\]

is a \((1 - \alpha)100\%\) confidence interval for \(\frac{1}{\theta}\).

4. Let \(X_1, X_2, ..., X_n\) be a random sample from a population with Laplace density function

\[
f(x; \theta) = \frac{1}{2\theta} e^{-\frac{|x|}{\theta}}, \quad -\infty < x < \infty
\]
where \(\theta\) is an unknown parameter. Show that

\[
\left[ \frac{2\sum_{i=1}^{n} |X_i|}{\chi^2_{1 - \frac{\alpha}{2}}(2n)}, \frac{2\sum_{i=1}^{n} |X_i|}{\chi^2_{\frac{\alpha}{2}}(2n)} \right]
\]

is a \((1 - \alpha)100\%\) confidence interval for \(\theta\).

5. Let \(X_1, X_2, ..., X_n\) be a random sample from a population with density function

\[
f(x; \theta) = \begin{cases} \frac{1}{2\theta^2} x^3 e^{-\frac{x^2}{2\theta}} & \text{for } 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}
\]
where \(\theta\) is an unknown parameter. Show that

\[
\left[ \frac{\sum_{i=1}^{n} X_i^2}{\chi^2_{1 - \frac{\alpha}{2}}(2n)}, \frac{\sum_{i=1}^{n} X_i^2}{\chi^2_{\frac{\alpha}{2}}(2n)} \right]
\]

is a \((1 - \alpha)100\%\) confidence interval for \(\theta\).

6. Let \(X_1, X_2, ..., X_n\) be a random sample from a population with density function

\[
f(x; \theta, \beta) = \begin{cases} \beta \theta \frac{x^{\beta-1}}{(1+x^\beta)^{\theta+1}} & \text{for } 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}
\]
where $\theta$ is an unknown parameter and $\beta > 0$ is a known parameter. Show that

$$\left[ \frac{\chi^2_\frac{1}{2}(2n)}{2 \sum_{i=1}^n \ln \left(1 + X_i^\beta\right)}, \frac{\chi^2_{1-\frac{1}{2}}(2n)}{2 \sum_{i=1}^n \ln \left(1 + X_i^\beta\right)} \right]$$

is a $(1 - \alpha)100\%$ confidence interval for $\theta$.

7. If $X_1, X_2, \ldots, X_n$ is a random sample from a population with density

$$f(x; \theta) = \begin{cases} \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}(x-\theta)^2} & \text{if } \theta \leq x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta$ is an unknown parameter, what is a $(1-\alpha)100\%$ approximate confidence interval for $\theta$ if the sample size is large?

8. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a distribution with a probability density function

$$f(x; \theta) = \begin{cases} (\theta + 1) x^{-\theta-2} & \text{if } 1 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \theta$ is a parameter. What is a $(1 - \alpha)100\%$ approximate confidence interval for $\theta$ if the sample size is large?

9. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a distribution with a probability density function

$$f(x; \theta) = \begin{cases} \theta^2 x e^{-\theta x} & \text{if } 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \theta$ is a parameter. What is a $(1 - \alpha)100\%$ approximate confidence interval for $\theta$ if the sample size is large?

10. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with density function

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-\frac{(x-4)}{\beta}} & \text{for } x > 4 \\ 0 & \text{otherwise,} \end{cases}$$
where $\beta > 0$. What is a $(1 - \alpha)100\%$ approximate confidence interval for $\theta$ if the sample size is large?

**GROUP B**

11. Five trials $X_1, X_2, \ldots, X_5$ of a Bernoulli experiment were conducted to test $H_0 : p = \frac{1}{2}$ against $H_a : p = \frac{3}{4}$. The null hypothesis $H_0$ will be rejected if $\sum_{i=1}^{5} X_i = 5$. Find the probability of Type I and Type II errors.

12. A manufacturer of car batteries claims that the life of his batteries is approximately normally distributed with a standard deviation equal to 0.9 year. If a random sample of 10 of these batteries has a standard deviation of 1.2 years, do you think that $\sigma > 0.9$ year? Use a 0.05 level of significance.

13. Let $X_1, X_2, \ldots, X_8$ be a random sample of size 8 from a Poisson distribution with parameter $\lambda$. Reject the null hypothesis $H_0 : \lambda = 0.5$ is the observed sum $\sum_{i=1}^{8} x_i \geq 8$. First, compute the significance level $\alpha$ of the test. Second, find the power function $\beta(\lambda)$ of the test as a sum of Poisson probabilities when $H_a$ is true.

14. Suppose $X$ has the density function

$$f(x) = \begin{cases} \frac{1}{\theta} & \text{for } 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

If one observation of $X$ is taken, what are the probabilities of Type I and Type II errors in testing the null hypothesis $H_0 : \theta = 1$ against the alternative hypothesis $H_a : \theta = 2$, if $H_0$ is rejected for $X > 0.92$.

15. Let $X$ have the density function

$$f(x) = \begin{cases} (\theta + 1) x^\theta & \text{for } 0 < x < 1 \text{ where } \theta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The hypothesis $H_0 : \theta = 1$ is to be rejected in favor of $H_1 : \theta = 2$ if $X > 0.90$. What is the probability of Type I error?
16. Let $X_1, X_2, ..., X_6$ be a random sample from a distribution with density function

$$f(x) = \begin{cases} \theta x^{\theta-1} & \text{for } 0 < x < 1 \text{ where } \theta > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The null hypothesis $H_0 : \theta = 1$ is to be rejected in favor of the alternative $H_a : \theta > 1$ if and only if at least 5 of the sample observations are larger than 0.7. What is the significance level of the test?

17. A researcher wants to test $H_0 : \theta = 0$ versus $H_a : \theta = 1$, where $\theta$ is a parameter of a population of interest. The statistic $W$, based on a random sample of the population, is used to test the hypothesis. Suppose that under $H_0$, $W$ has a normal distribution with mean 0 and variance 1, and under $H_a$, $W$ has a normal distribution with mean 4 and variance 1. If $H_0$ is rejected when $W > 1.50$, then what are the probabilities of a Type I or Type II error respectively?

18. Let $X_1$ and $X_2$ be a random sample of size 2 from a normal distribution $N(\mu, 1)$. Find the best likelihood ratio critical region of size 0.005 for testing the null hypothesis $H_0 : \mu = 0$ against the composite alternative $H_a : \mu \neq 0$?

19. Suppose $X_1, X_2, ..., X_{10}$ be a random sample from a Poisson distribution with mean $\theta$. What is the most powerful (or best ) critical region of size 0.08 for testing the null hypothesis $H_0 : \theta = 0.1$ against $H_a : \theta = 0.5$?

20. Let $X$ be a random sample of size 1 from a distribution with probability density function

$$f(x, \theta) = \begin{cases} (1 - \frac{\theta}{2}) + \theta x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For a significance level $\alpha = 0.1$, what is the best critical region for testing the null hypothesis $H_0 : \theta = -1$ against $H_a : \theta = 1$?
21. Let $X_1, X_2$ be a random sample of size 2 from a distribution with probability density function

$$f(x, \theta) = \begin{cases} \frac{\theta^x e^{-\theta}}{x!} & \text{if } x = 0, 1, 2, 3, \ldots \\ 0 & \text{otherwise}, \end{cases}$$

where $\theta \geq 0$. For a significance level $\alpha = 0.053$, what is the best critical region for testing the null hypothesis $H_0 : \theta = 1$ against $H_a : \theta = 2$? Sketch the graph of the best critical region.

22. Let $X_1, X_2, \ldots, X_8$ be a random sample of size 8 from a distribution with probability density function

$$f(x, \theta) = \begin{cases} \frac{\theta^x e^{-\theta}}{x!} & \text{if } x = 0, 1, 2, 3, \ldots \\ 0 & \text{otherwise}, \end{cases}$$

where $\theta \geq 0$. What is the best likelihood ratio critical region for testing the null hypothesis $H_0 : \theta = 1$ against $H_a : \theta \neq 1$? If $\alpha = 0.1$ can you determine the best likelihood ratio critical region?

23. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from a distribution with probability density function

$$f(x, \theta) = \begin{cases} \frac{x^6 e^{\frac{5}{1(\theta)^7}}}{1(\theta)^7} & \text{if } x > 0 \\ 0 & \text{otherwise}, \end{cases}$$

where $\theta \geq 0$. What is the best likelihood ratio critical region for testing the null hypothesis $H_0 : \beta = 5$ against $H_a : \beta \neq 5$? What is the most powerful test?

24. Let $X_1, X_2, \ldots, X_5$ denote a random sample of size 5 from a population $X$ with probability density function

$$f(x; \theta) = \begin{cases} (1 - \theta)^x \theta & \text{if } x = 1, 2, 3, \ldots, \infty \\ 0 & \text{otherwise}, \end{cases}$$
where \(0 < \theta < 1\) is a parameter. What is the likelihood ratio critical region for testing \(H_0 : \theta = 0.5\) versus \(H_a : \theta \neq 0.5\)?

25. Let \(X_1, X_2, X_3\) denote a random sample of size 3 from a population \(X\) with probability density function
\[
f(x; \mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2}} \quad -\infty < x < \infty,
\]
where \(-\infty < \mu < \infty\) is a parameter. What is the likelihood ratio critical region for testing \(H_0 : \mu = 3\) versus \(H_a : \mu \neq 3\)?

26. Let \(X_1, X_2, X_3\) denote a random sample of size 3 from a population \(X\) with probability density function
\[
f(x; \theta) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & \text{if } 0 < x < \infty \\ 0 & \text{otherwise}, \end{cases}
\]
where \(0 < \theta < \infty\) is a parameter. What is the likelihood ratio critical region for testing \(H_0 : \theta = 3\) versus \(H_a : \theta \neq 3\)?

27. Let \(X_1, X_2, X_3\) denote a random sample of size 3 from a population \(X\) with probability density function
\[
f(x; \theta) = \begin{cases} \frac{e^{-\theta \theta^x}}{x!} & \text{if } x = 0, 1, 2, 3, \ldots, \infty \\ 0 & \text{otherwise}, \end{cases}
\]
where \(0 < \theta < \infty\) is a parameter. What is the likelihood ratio critical region for testing \(H_0 : \theta = 0.1\) versus \(H_a : \theta \neq 0.1\)?

28. A box contains 4 marbles, \(\theta\) of which are white and the rest are black. A sample of size 2 is drawn to test \(H_0 : \theta = 2\) versus \(H_a : \theta \neq 2\). If the null hypothesis is rejected if both marbles are the same color, find the significance level of the test.

29. Let \(X_1, X_2, X_3\) denote a random sample of size 3 from a population \(X\) with probability density function
\[
f(x; \theta) = \begin{cases} \frac{1}{\theta} & \text{for } 0 \leq x \leq \theta \\ 0 & \text{otherwise}, \end{cases}
\]
where $0 < \theta < \infty$ is a parameter. What is the likelihood ratio critical region of size $\frac{117}{125}$ for testing $H_0 : \theta = 5$ versus $H_a : \theta \neq 5$?

30. Let $X_1, X_2$ and $X_3$ denote three independent observations from a distribution with density

$$f(x; \beta) = \begin{cases} \frac{1}{\beta} e^{-\frac{x}{\beta}} & \text{for } 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \beta < \infty$ is a parameter. What is the best critical region of size $0.025$ for testing $H_0 : \beta = 5$ versus $H_a : \beta = 10$?