Homework 3  September 9, 2005  Math 521

Direction: This homework is due on September 16, 2005. In order to receive full credit, answer each problem completely and must show all work.

1. With picture and words, describe each symmetry in $D_3$ (the set of symmetries of an equilateral triangles). Write out a complete multiplication table for $D_3$.

Answer: The following actions leave the triangle looking like unchanged.

![Symmetry Diagrams](image)

The set

$$G = \{ R_0, R_1, R_2, F_1, F_2, F_3 \}$$

together with function composition $\circ$ forms a symmetry group for an equilateral triangles. This symmetry group is called the dihedral group of order 6 and denoted by $D_3$. The multiplication table of $D_3$ is

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</table>
2. Describe the symmetry group of the logo and write out a complete multiplication table for this group.

**Answer:** The symmetry group of the given logo consists of only three rotations: 0° degree rotation, 120° rotation and 240° rotation about the center. Let these rotations be \( R_0, R_1 \) and \( R_2 \), respectively. The set
\[
G = \{ R_0, R_1, R_2 \}
\]
together with function composition \( \circ \) forms a symmetry group of this logo. The Cayley table for this symmetry group is given below:

\[
\begin{array}{c|ccc}
\circ & R_0 & R_1 & R_2 \\
\hline
R_0 & R_0 & R_1 & R_2 \\
R_1 & R_1 & R_2 & R_0 \\
R_2 & R_2 & R_0 & R_1 \\
\end{array}
\]

3. Show that \( \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \) does not have a multiplicative inverse in \( GL(2, \mathbb{R}) \).

**Answer:** The determinant of the matrix \( \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \) is
\[
det \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = 2 - 2 = 0.
\]

Hence it does not have an inverse in \( GL(2, \mathbb{R}) \).

4. Show that the group \( GL(2, \mathbb{R}) \) is non-Abelian, by finding a pair of matrices \( A \) and \( B \) in \( GL(2, \mathbb{R}) \) such that \( AB \neq BA \).

**Answer:** Let \( A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Then
\[
AB = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}
\]
and
\[
BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.
\]

Hence \( AB \neq BA \) and \( GL(2, \mathbb{R}) \) is nonabelian group.
5. Find the inverse of the element \( \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix} \) in \( GL(2, \mathbb{Z}_{11}) \).

**Answer:** The determinant of the matrix \( \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix} \) is
\[
det \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix} = 10 - 18 = -8 = 3 \mod 11.
\]
The multiplicative inverse of 3 is 4 since \( 3 \cdot 4 = 1 \mod 11 \). The inverse of \( \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix} \) is given by
\[
\begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} 5 & -6 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 5 \cdot 4 & -6 \cdot 4 \\ -3 \cdot 4 & 2 \cdot 4 \end{pmatrix} = \begin{pmatrix} 9 & -2 \\ -1 & 8 \end{pmatrix} = \begin{pmatrix} 9 & 9 \\ 10 & 8 \end{pmatrix}.
\]

6. Let \( G \) be a group such that whenever \( ab = ca \) for arbitrary \( a, b, c \in G \), then \( b = c \). Show that such a group \( G \) is Abelian.

**Answer:** Let \( x, y \in G \). We want to show that \( G \) is abelian, that is \( xy = yx \) for all \( x, y \in G \).
Take \( a = x \) and \( b = yx \) and \( c = xy \). Then
\[
ab = x(yx) = (xy)x = ca
\]
Hence we get \( b = c \) and this implies
\[
yx = xy
\]
for all \( x, y \in G \). Therefore \( G \) is abelian.

7. Let \( a \) and \( b \) be any two elements of an Abelian group. Prove that \( (ab)^4 = a^4 b^4 \).

**Answer:** Suppose the group is abelian, that is \( ab = ba \) for all \( a, b \in G \).

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= a^4 b^4.
\]
8. Prove that a group $G$ is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

**Answer:** Suppose $G$ is abelian, that is $ab = ba$ for all $a, b \in G$. We want to show that $(ab)^{-1} = a^{-1}b^{-1}$. Since
\[(ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} \quad \text{(because G abelian)}\]
hence we have $(ab)^{-1} = a^{-1}b^{-1}$.

Next assume $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. We will show that $G$ is abelian. Since
\[(ab)^{-1} = a^{-1}b^{-1} \quad a (ab)^{-1} = a a^{-1}b^{-1} \quad ba (ab)^{-1} = bb^{-1} \quad ba (ab)^{-1} = e \quad ba (ab)^{-1} (ab) = ab \quad ba = ab, \]
hence $G$ is abelian.

9. Prove that if $(ab)^2 = a^2b^2$ in a group $G$, then $ab = ba$.

**Answer:** Since
\[(ab)^2 = a^2 b^2 \quad ab ab = aa bb \quad bab = abb \quad \text{(by cancelling a from left)} \quad ba = ab \quad \text{(by cancelling b from right)}, \]
hence $G$ is abelian.

10. Let $a, b$ and $c$ be elements of a group. Solve the equation $axb = c$ for $x$.

**Answer:** $x = a^{-1} c b^{-1}$. 