Direction: This homework is due on September 30, 2005. In order to receive full credit, answer each problem completely and must show all work.

1. Prove that in any group, an element and its inverse have the same order.

Answer: Let a be an element of group G and let $a^{-1}$ be the inverse of a. If $|a|$ is infinite, then $|a^{-1}|$ is also infinite since $(a^n)^{-1} = (a^{-1})^n$. Suppose $|a| = n < \infty$ and $|a^{-1}| = m < \infty$ where m and n are positive integers. We want to show that $m = n$. Since $(a^n)^{-1} = (e)^{-1} = e$, we see that m divide n, that is, $m/n$. Similarly, since $a^m = (a^{-m})^{-1} = (e)^{-1} = e$, we see that n divide m, that is, $n/m$. Therefore $m/n$ and $n/m$ imply $m = n$. Hence the order of a and $a^{-1}$ are same.

2. Show that $U(14) = \langle 3 \rangle = \langle 5 \rangle$. Is $U(14) = \langle 11 \rangle$?

Answer: The set of elements of $U(14)$ is given by $\{ 1, 3, 5, 9, 11, 13 \}$. The subgroup generated by $\langle 3 \rangle$ is given by

$$< 3 > = \{ 3^k | k \in \mathbb{Z} \} = \{ 3, 9, 13, 11, 5, 1 \}.$$  

and

$$< 5 > = \{ 5^k | k \in \mathbb{Z} \} = \{ 5, 11, 13, 9, 3, 1 \}.$$  

Hence $U(14) = \langle 3 \rangle = \langle 5 \rangle$. The subgroup generated by $\langle 11 \rangle$ is given by

$$< 11 > = \{ 11^k | k \in \mathbb{Z} \} = \{ 11, 9, 1 \} \neq U(14).$$

3. Let G be a group. Show that $Z(G) = \bigcap_{a \in G} C(a)$.

Answer: Let G be a group. The $Z(G)$ is defined as $Z(G) = \{ a \in G | ax = xa \ \forall x \in G \}$ and the centralizer of the element a is given by $C(a) = \{ g \in G | ga = ag \}$. Let $x \in Z(G)$. Then $ax = xa$ for all a in G. This implies that $x \in C(a)$ for all $a \in G$. Hence $x \in \bigcap_{a \in G} C(a)$. Therefore $Z(G) \subseteq \bigcap_{a \in G} C(a)$. Next, let $x \in \bigcap_{a \in G} C(a)$. Then $x \in C(a)$ for all $a \in G$. This implies that $ax = xa$ for all $a \in G$. Hence $x \in Z(G)$, and $\bigcap_{a \in G} C(a) \subseteq Z(G)$. So we have shown that $Z(G) = \bigcap_{a \in G} C(a)$. 
4. Find the center of the group $GL(2, \mathbb{R})$ under matrix multiplication.

**Answer:** The group $GL(2, \mathbb{R})$ is given by

$$GL(2, \mathbb{R}) = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} \mid xw - yz \neq 0, \ x, y, z, w \in \mathbb{R} \right\}$$

and $Z(GL(2, \mathbb{R})) = \left\{ A \in GL(2, \mathbb{R}) \mid AX = XA \ \forall X \in GL(2, \mathbb{R}) \right\}$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Then $AX = \begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix}$ and $XA = \begin{pmatrix} ax + cy & bx + dy \\ az + cw & bz + dw \end{pmatrix}$. From the relation $AX =XA$ for all $X$ in $GL(2, \mathbb{R})$, we get

$$\begin{pmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{pmatrix} = \begin{pmatrix} ax + cy & bx + dy \\ az + cw & bz + dw \end{pmatrix}$$

for all $x, y, z, w \in \mathbb{R}$. Hence we obtain the following four equations: (1) $ax + bz = ax + cy$, (2) $ay + bw = bx + dy$, (3) $cx + dz = az + cw$, and (4) $cy + dw = bz + dw$ for all $x, y, z, w \in \mathbb{R}$. Solving the first equation with $z = y = 1$, we get $b = c$. Again from first equation with $z = 2$, $y = 1$, we obtain $2b = c$. Therefore $b = 0$ and hence $c = 0$. Using $b = 0 = c$ in the second equation, we obtain $a = d$. Hence $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ where $a \neq 0$. Thus the center of the group $GL(2, \mathbb{R})$ is given by

$$Z(GL(2, \mathbb{R})) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{R} \text{ with } a \neq 0 \right\}.$$

5. Let $x$ be an element of a group $G$. Suppose $|x| = 6$. Find $|x^2|, |x^3|, |x^4|, |x^5|$.

**Answer:** We are given that $|x| = 6$. Since $(x^2)^3 = x^6 = e$, the order of $x^2$ is 3. Similarly, since $(x^3)^2 = x^6 = e$, the order of $x^3$ is 2. Note that $(x^4)^3 = x^{12} = (x^6)^2 = e$. Hence the order of $x^4$ is 3. Similarly, $(x^5)^6 = x^{30} = (x^6)^5 = e$. Therefore the order of $x^5$ is 6.

6. Show that if $G$ is a group, the $Z(G)$ is a subgroup of $G$.

**Answer:** Since the identity $e$ of the group $G$ commutes with every element of $G$, the identity $e$ belongs to $Z(G)$. Hence $Z(G) \neq \emptyset$. Let $a, b \in Z(G)$ be any two arbitrary elements. Then $ax = xa$ and $bx = xb$ for all $x \in G$. Since

$$abx = a(bx) = a(xb) = (ax)b = (xa)b = xab,$$

the product $ab \in Z(G)$ whenever $a, b \in Z(G)$. Similarly, let $a \in Z(G)$. Since

$$x a^{-1} a x a^{-1} = x a^{-1} (a x) a^{-1} = x a^{-1} (x a) a^{-1} = a^{-1} x,$$

$a^{-1} \in Z(G)$ whenever $a \in Z(G)$. Hence by two-step subgroup test $Z(G) \leq G$. 


7. What is the Cayley table of the dihedral group $D_3$? Find the center of this dihedral group. Also, find the centralizer of the element $R_{120}$ (that is, rotation by 120°) in $D_3$.

**Answer:** Let $R_0$, $R_1$, and $R_2$ be the rotations of an equilateral triangles about the center by 0°, 120°, and 240°, respectively. From Homework 3, the Cayley table of $D_3$ is

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Since only the row and column headed by element $R_0$ are identical, the center of the group $D_3$ is given by $Z(D_3) = \{ R_0 \}$. The centralizer of the element $R_1$ is $C(R_1) = \{ R_0, R_1, R_2 \}$.

8. Find the center of the group $G$ whose Cayley table is given below.

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**Answer:** Since the rows and columns headed by the elements 1 and 5 are identical, therefore the center of the group $G$ is $Z(G) = \{ 1, 5 \}$.

9. For the group in problem 8, find the centralizer for each element of $G$. What is $\bigcap_{a \in G} C(a)$?

**Answer:** The centralizer of the each element of $G$ are:

- $C(1) = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$, $C(2) = \{ 1, 2, 5, 6 \}$, $C(3) = \{ 1, 3, 5, 7 \}$
- $C(4) = \{ 1, 4, 5, 8 \}$, $C(5) = \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$, $C(6) = \{ 1, 2, 5, 6 \}$
- $C(7) = \{ 1, 3, 5, 7 \}$, $C(8) = \{ 1, 4, 5, 8 \}$

For this group $G$, the $\bigcap_{a \in G} C(a)$ is

$$\bigcap_{a \in G} C(a) = \{ 1, 5 \}.$$
10. Show that for each $x$ in the group $G$, the centralizer $C(x)$ of $x$ is a subgroup of $G$.

**Answer:** Let $x$ be an arbitrary element of $G$.

Since the identity $e$ of the group $G$ commutes with every element of $G$, therefore it commutes with $x$. Hence $e \in C(x)$ and $C(x) \neq \emptyset$. Let $a, b \in C(x)$ be any two arbitrary elements. Then by definition of $C(x)$, we have $ax = xa$ and $bx = xb$. Since

$$abx = a(bx) = a(xb) = (ax)b = (xa)b = xab,$$

the product $ab \in C(x)$ whenever $a, b \in C(x)$. Similarly, let $a \in C(x)$. Since

$$x a^{-1} = a^{-1} a x a^{-1} = a^{-1} (ax) a^{-1} = a^{-1} (xa) a^{-1} = a^{-1} x,$$

$a^{-1} \in C(x)$ whenever $a \in C(x)$. Hence by two-step subgroup test $C(x) \leq G$. 