Direction: This homework is due on October 14, 2005. In order to receive full credit, answer each problem completely and must show all work.

1. For the following permutations $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 5 & 9 & 1 & 8 & 2 & 7 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 6 & 5 & 3 & 4 & 6 & 8 & 2 & 7 \end{pmatrix}$, find (a) $\alpha \beta$, (b) $\beta \alpha$, (c) $\alpha^{-1}$, (d) $\beta^{-1}$, (e) $\alpha^{-1} \beta^{-1}$, (f) $\beta^{-1} \alpha^{-1}$.

Answer: (a) $\alpha \beta = (1, 3, 5)(2, 8)(6, 7)(4, 9)$, (b) $\beta \alpha = (1, 3, 5)(2, 8)(6, 7)(4, 9)$, (c) $\alpha^{-1} = (1, 5, 3)(2, 7, 8, 6)(4, 9)$, (d) $\beta^{-1} = (2, 7, 8, 6)$, (e) $\alpha^{-1} \beta^{-1} = (1, 5, 3)(2, 8)(4, 9)(6, 7)$, and (f) $\beta^{-1} \alpha^{-1} = (1, 5, 3)(2, 8)(4, 9)(6, 7)$.

2. Find the order of each of the following permutations:
   (a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 5 & 9 & 1 & 8 & 2 & 7 & 4 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 6 & 3 & 4 & 5 & 8 & 2 & 7 & 9 \end{pmatrix}$.

Answer: (a) The order of $(1, 3, 5)(2, 6, 8, 7)(4, 9)$ is $\text{lcm}(3, 4, 2) = 12$. (b) The order of $(2, 6, 8, 7)$ is 4.

3. What is the order of each of the following permutations?
   (a) $(1, 2, 4)(3, 5)$, (b) $(1, 2, 4)(3, 5, 6)$, (c) $(1, 2, 4)(3, 5)$, (d) $(1, 2, 4)(3, 5, 7, 8)$.

Answer: (a) $\text{lcm}(3, 3) = 3$, (b) $\text{lcm}(3, 3) = 3$, (c) $\text{lcm}(3, 2) = 6$, and (d) $\text{lcm}(3, 4) = 12$.

4. If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 5 & 4 & 7 & 6 & 8 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{pmatrix}$, then express these permutations $\alpha$ and $\beta$ as products of disjoint cycles. Also express $\alpha$ and $\beta$ as products of 2-cycles.

Answer: $\alpha = (1, 2)(4, 5)(6, 7)$, and $\beta = (2, 7)(2, 4)(2, 8)(2, 3)(5, 6)$.

5. Determine whether the following permutations are even or odd. (a) $(1, 3, 5)$, (b) $(1, 3, 5, 6)$, (c) $(1, 3, 5, 6, 7)$, (d) $(1, 2)(1, 3, 4)(1, 5, 2)$, (e) $(1, 2, 3, 4)(3, 5, 2, 1)$.

Answer: (a) even, (b) odd, (c) even, (d) $(1, 5)(2, 4)(2, 3)$ odd, and (e) $(1, 4)(3, 5)$ even.
6. If \( f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 5 & 4 & 7 & 6 & 8 \end{pmatrix} \), the find \( f^2, f^3, f^4, f^5, \ldots, f^8 \). Also find \( f^{-2}, f^{-3}, f^{-4}, f^{-5}, \ldots, f^{-8} \).

**Answer:** Since \( f = (1, 2)(4, 5)(6, 7) \), the order of \( f \) is the \( \text{lcm}(2, 2, 2) \) which is 2. Hence \( f^2 = (1) \). Using this relation, we obtain \( f^3 = f \), \( f^4 = (f^2)^2 = (1) \), \( f^5 = f \), etc. Similarly, since order of \( f \) is 2, we have \( f = f^{-1} \). Hence \( f^{-2} = (f^{-1})^2 = (1) \), \( f^{-3} = f \), \( f^{-4} = (1) \), \( f^{-5} = f \) etc.

7. Compute \( gfg^{-1} \) for each pair \( f, g \).

(a) \( f = (1, 4, 2, 3); g = (1, 3, 2) \), (b) \( f = (1, 2, 4, 6); g = (2, 5, 4, 6) \),

(c) \( f = (1, 3, 5, 2)(4, 6); g = (1, 3, 6)(2, 4, 5) \).

**Answer:** Observe that if \( i_1 \) and \( i_2 \) are two positive integers such that \( f(i_1) = i_2 \), then \( gfg^{-1}(g(i_1)) = g(i_2) \). This means that if \( (i_1, i_2, i_3, \ldots, i_n) \) is one disjoint cycle in \( f \), then \( (g(i_1), g(i_2), g(i_3), \ldots, g(i_n)) \) is the corresponding disjoint cycle in \( gfg^{-1} \). We use this observation to solve this problem in an easy manner.

(a) Since \( f = (1, 4, 2, 3) \), the corresponding \( gfg^{-1} \) is

\[
gfg^{-1} = (g(1), g(4), g(2), g(3)) = (1, 2, 3, 4).
\]

(b) Since \( f = (1, 2, 4, 6) \), the corresponding \( gfg^{-1} \) is

\[
gfg^{-1} = (g(1), g(2), g(4), g(6)) = (1, 5, 6, 2).
\]

(c) Since \( f = (1, 3, 5, 2)(4, 6) \), the corresponding \( gfg^{-1} \) is

\[
gfg^{-1} = (g(1), g(3), g(5), g(2))(g(4), g(6)) = (3, 6, 2, 4)(5, 1) = (2, 4, 3, 6)(1, 5).
\]

8. For a given permutations \( f \) and \( h \), find a \( g \) such that \( gfg^{-1} = h \).

(a) \( f = (1, 5, 3); h = (2, 6, 4) \), and (b) \( f = (1, 2, 5, 7); h = (3, 4, 6, 8) \).

**Answer:** (a) By our observation in Problem 7, if \( f = (1, 5, 3) \), then \( gfg^{-1} = (g(1), g(5), g(3)) \). We want this to be equal to \( h = (2, 6, 4) \). Hence

\[
gfg^{-1} = (g(1), g(5), g(3)) (g(2)) (g(4)) (g(6)) = (2, 6, 4)(1)(3)(5) = h.
\]

Hence by comparison, we get \( g(1) = 2, g(2) = 1, g(3) = 4, g(4) = 3, g(5) = 6, g(6) = 5 \).

Hence \( g \) is given by \( g = (1, 2)(3, 4)(5, 6) \).

(b) Similarly, \( g = (1, 3)(2, 4)(5, 6)(7, 8) \).
9. Let \( G = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \) be a group whose multiplication table is given by

\[
\begin{array}{cccccccccccc}
\times & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 1 & 2 & 3 & 4 & 0 & 6 & 7 & 8 & 9 & 5 \\
2 & 2 & 3 & 4 & 0 & 1 & 7 & 8 & 9 & 5 & 6 \\
3 & 3 & 4 & 0 & 1 & 2 & 8 & 9 & 5 & 6 & 7 \\
4 & 4 & 0 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 \\
5 & 5 & 9 & 8 & 7 & 6 & 0 & 4 & 3 & 2 & 1 \\
6 & 6 & 5 & 9 & 8 & 7 & 1 & 0 & 4 & 3 & 2 \\
7 & 7 & 6 & 5 & 9 & 8 & 2 & 1 & 0 & 4 & 3 \\
8 & 8 & 7 & 6 & 5 & 9 & 3 & 2 & 1 & 0 & 4 \\
9 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
\end{array}
\]

For the subgroup \( H = \{0, 1, 2, 3, 4\} \) find (a) \( 2H2^{-1} \), (b) \( 3H3^{-1} \), and (c) \( 4H4^{-1} \). What is the multiplication table of the subgroup \( 3H3^{-1} \)?

**Answer:** (a) Since \( 2^{-1} = 3 \), we have

\( 2H2^{-1} = 2H3 = \{2 \times 0 \times 3, 2 \times 1 \times 3, 2 \times 2 \times 3, 2 \times 3 \times 3, 2 \times 4 \times 3\} = \{0, 1, 2, 3, 4\} = H. \)

(b) Since \( 3^{-1} = 2 \), we have

\( 3H3^{-1} = 3H2 = \{3 \times 0 \times 2, 3 \times 1 \times 2, 3 \times 2 \times 2, 3 \times 3 \times 2, 3 \times 4 \times 2\} = \{0, 1, 2, 3, 4\} = H. \)

(c) Since \( 4^{-1} = 1 \), we have

\( 4H4^{-1} = 4H1 = \{4 \times 0 \times 1, 4 \times 1 \times 1, 4 \times 2 \times 1, 4 \times 3 \times 1, 4 \times 4 \times 1\} = \{0, 1, 2, 3, 4\} = H. \)

The multiplication table of \( 3H3^{-1} \) is same as \( H \) since \( 3H3^{-1} = H \). Hence

\[
\begin{array}{cccccccccccc}
\times & 0 & 1 & 2 & 3 & 4 \\
0 & 0 & 1 & 2 & 3 & 4 \\
1 & 1 & 2 & 3 & 4 & 0 \\
2 & 2 & 3 & 4 & 0 & 1 \\
3 & 3 & 4 & 0 & 1 & 2 \\
4 & 4 & 0 & 1 & 2 & 3 \\
\end{array}
\]

10. For \( G \) and \( H \) in problem 9, find the set \( N(H) = \{x \in G \mid xHx^{-1} = H\} \).

**Answer:** From Problem 9, we know that \( 2H2^{-1} = H, 3H3^{-1} = H, \) and \( 4H4^{-1} = H. \)

Similarly, we can verify that \( 0H0^{-1} = H \) and \( 1H1^{-1} = H. \) From the multiplication table for \( G \), we see that \( 5^{-1} = 5, 6^{-1} = 6, 7^{-1} = 7, 8^{-1} = 8, \) and \( 9^{-1} = 9. \) Hence if \( x = 5, 6, 7, 8, 9, \) we can easily show that \( xHx^{-1} = xHx = \{0, 1, 2, 3, 4\} = H. \) Therefore

\( N(H) = \{x \in G \mid xHx^{-1} = H\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} = G. \)